Uniform convergence of semigroups of analytic functions

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Abstract: Let Ω be a region in the complex plane $\mathbb{C}$ and let $\{\phi_t\}_{t\geq 0}$ be a continuous semigroup of functions on Ω; that is, $\phi_t \colon \Omega \to \Omega$ is holomorphic for every $t \geq 0$, $\phi_0(z) = z$, for every $z \in \Omega$, $\phi_t \circ \phi_s = \phi_{s+t}$, for every $s, t \geq 0$, and

$$\phi_t(z) \to z\,, \qquad \text{as } t \text{ goes to } 0^+, \tag{1}$$

uniformly on compact subsets of Ω. Despite the definition of continuous semigroup only requires in (1) the uniform convergence on compact subsets, P. Gumenyuk has proved, using the No Koebe Arcs Theorem, that, in the case of the unit disc ($\Omega = \mathbb{D}$), the convergence in (1) is uniform on the whole $\mathbb{D}$. In the case where Ω is a half-plane it is easy to give examples of semigroups where the convergence in (1) is not uniform on Ω.

The aim of this talk is to present some recent results obtained in collaboration with C. Gómez-Cabello and L. Rodríguez-Piazza about uniform convergence for the disc and the half-plane. We provide an improvement of Gumenyuk's result showing that for every semigroup $\{\phi_t\}_{t\geq 0}$ on $\mathbb{D}$ we have

$$\sup_{z\in\mathbb{D}} |\phi_t(z) - z| = O(\sqrt{t}), \quad t \to 0^+.$$

Examples show that $O(\sqrt{t})$ is the best possible ratio of uniform convergence valid for all semigroups on $\mathbb{D}$.

If Ω is a half-plane we prove that there is uniform convergence in (1) under certain boundedness conditions on the infinitesimal generator of the semi group. These boundedness conditions are fulfilled when the semigroup $\{\phi_t\}_{t\geq 0}$ is included in the Gordon-Hedenmalm class (the one which produces bounded composition operators on the Hardy space of Dirichlet series).

An important ingredient in the proofs of these results is the use of harmonic measures, which we have done through a classic result of M. Lavrentiev.