Coadjoint orbits relevant to ideal fluid dynamics

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Plan of the talk

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5. Vortex sheets

1. Introduction

Ideal fluid flow

Riemannian manifold (M,g) with induced volume form μ and Levi-Civita connection ∇.

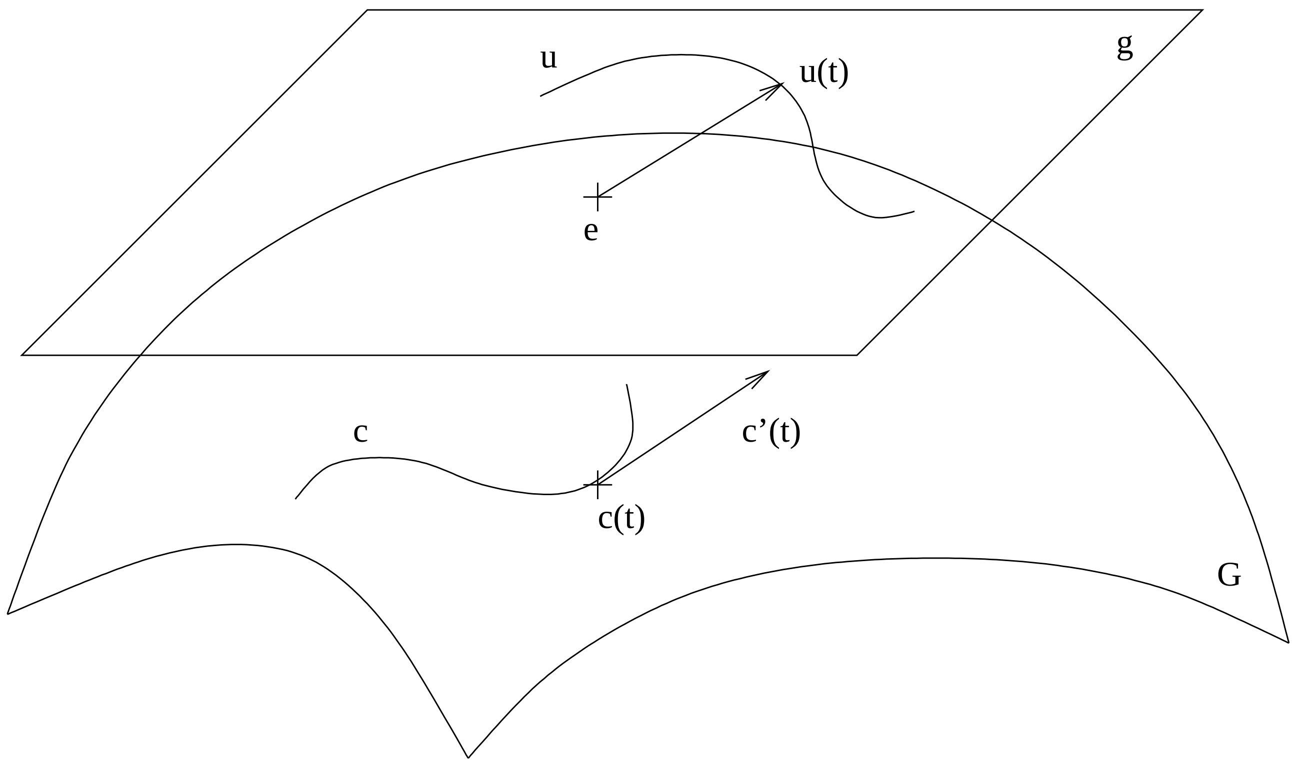
The geodesic equation on the group $G = \mathrm{Diff}_{\mathrm{vol}}(M)$ of volume preserving diffeomorphisms of a Riemannian manifold (M,g) for the right invariant L^2-metric

$$\langle u,v\rangle = \int_M g(u,v)\mu, \qquad u,v \in \mathfrak{g} = \mathfrak{X}_{\mathrm{vol}}(M)$$

is Euler equation for ideal fluid flow with velocity u and pressure p:

$$\partial_t u = -\nabla_u u - \operatorname{grad} p, \qquad \operatorname{div} u = 0.$$

Allowed: M noncompact or with boundary



Geodesic on a Lie group with right invariant Riemannian metric

Vorticity

The vorticity 2-form $\sigma = du^\flat \in d\Omega^1(M)$ satisfies Helmholtz equation

$$\partial_t \sigma = -L_u \sigma,$$

hence it is driven by the flow. The vorticity is $\operatorname{curl} u$ in 3D resp. the function σ/μ in 2D.

The smooth dual of $\mathfrak{g} = \mathfrak{X}_{\text{vol}}(M)$ is $\mathfrak{g}^*_{\text{reg}} = \Omega^1(M)/d\Omega^0(M)$:

$$([\gamma], u) = \int_M \gamma(u)\mu$$

If $H^1(M) = 0$, it is the space of vorticities $\mathfrak{g}^*_{\text{reg}} = d\Omega^1(M)$.
The coadjoint action is the natural one, by pullback, so the vorticity 2-form is confined to a coadjoint orbit:

$$\sigma(t) = \operatorname{Ad}^*_{\varphi(t)} \sigma(0).$$

Lie-Poisson equations

The symplectic leaves of the Lie-Poisson bracket on $\mathfrak{g}^*$

$$\{f, g\}(\sigma) = \left\langle \sigma, \left[\frac{\delta f}{\delta \sigma}, \frac{\delta g}{\delta \sigma}\right]\right\rangle, \qquad f, g \in C^\infty(\mathfrak{g}^*).$$

are the coadjoint orbits endowed with the Kirilov-Kostant-Souriau symplectic form

$$\omega_{KKS}(\mathrm{ad}^*_\xi \sigma, \mathrm{ad}^*_\eta \sigma) = \langle \sigma, [\xi, \eta]\rangle.$$

The Lie-Poisson equations for $H \in C^\infty(\mathfrak{g}^*)$

$$\tfrac{d}{dt}\sigma - \mathrm{ad}^*_{\frac{\delta H}{\delta \sigma}} \sigma = 0$$

restrict to symplectic Hamiltonian systems on each coadjoint orbit.

Reeb graphs

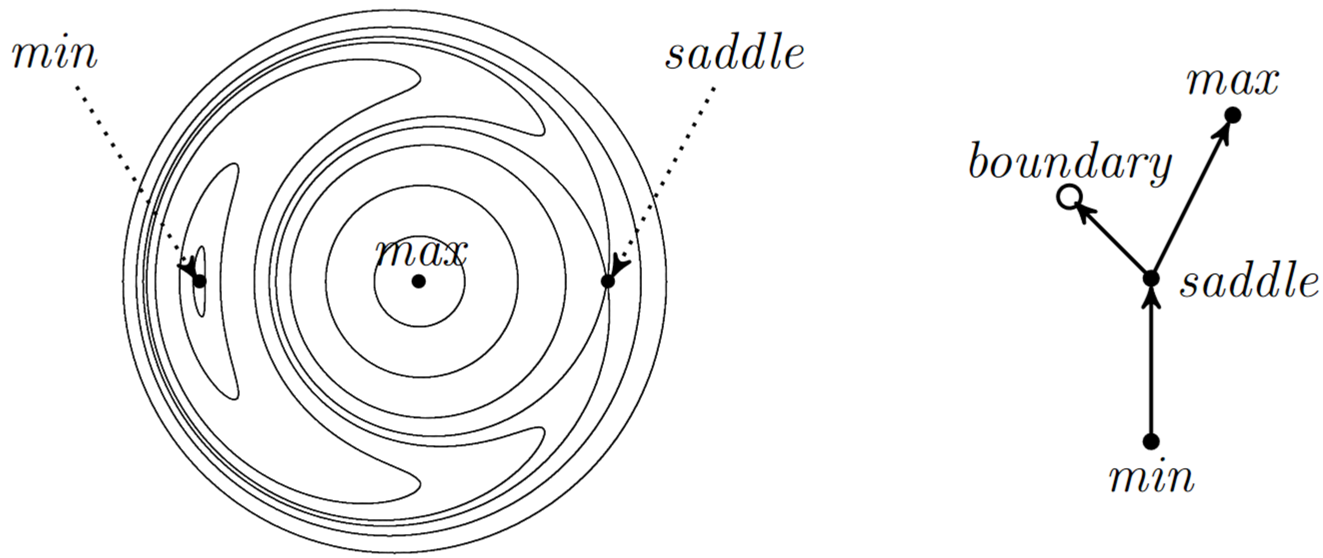


Figure 2: A simple Morse function on a disk and the associated graph.

[Izosimov-Khesin'17] Classify coadjoint orbits in the smooth dual in the 2D setting

Singular vorticities

Singular vorticities in $\mathfrak{g}^*$, that are supported on submanifolds of M, are expected to live on coadjoint orbits [Marsden-Weinstein'83].

1. codimension 2:
   - point vortices in 2D
   - vortex filaments in 3D

2. codimension 1:
   - vortex loops in 2D
   - vortex sheets in 3D

Coadjoint orbits and momentum maps

The defining identity of the momentum map $J : M \to \mathfrak{g}^*$ for a Hamiltonian action is:

$$d\langle J, X\rangle = i_{\zeta_X}\Omega, \quad X \in \mathfrak{g}$$

The inclusion $\mathcal{O} \subset \mathfrak{g}^*$ is a momentum map for the coadjoint action of G on the coadjoint orbit $(\mathcal{O}, \omega_{KKS})$.

Lemma: Let G act on a symplectic manifold (M, Ω) by a Hamiltonian action with injective and equivariant momentum map $J : M \to \mathfrak{g}^*$. If the action is transitive, then $J(M) = \mathcal{O}$ is a coadjoint orbit and $J^*\omega_{KKS} = \Omega$.

2. Point vortices

The Hamiltonian group of $\mathbb{R}^2$

Let $\mathbb{R}^2$ be endowed with the standard volume form ω and let G be the group of compactly supported Hamiltonian diffeomorphisms

$$G = \mathrm{Ham}_c(\mathbb{R}^2)$$

with corresponding Lie algebra

$$\mathfrak{g} = \mathfrak{X}_{\mathrm{ham,c}}(\mathbb{R}^2) = C_c^\infty(\mathbb{R}^2)$$

consisting of Hamiltonian vector fields X_h on $\mathbb{R}^2$.

Point vortices

Let $\mathsf{Conf}_N(\mathbb{R}^2) = (\mathbb{R}^2)^N \backslash \Delta^N$ be the configuration space of N points in $\mathbb{R}^2$, where the fat diagonal is

$$\Delta^N = \{(x_1, ..., x_N) \in (\mathbb{R}^2)^N : x_i = x_j \quad \text{for some} \quad i \neq j\},$$

endowed with the symplectic form $\sum_{i=1}^N \Gamma_i \, \mathrm{pr}_i^* \, \omega$.

The momentum map for the action of $G = \mathsf{Ham}_c(\mathbb{R}^2)$

$$J : \mathsf{Conf}_N(\mathbb{R}^2) \to \mathfrak{g}^* = C_c^\infty(\mathbb{R}^2)^*, \qquad J(x_1, ..., x_N) = \sum_{i=1}^N \Gamma_i \delta_{x_i}$$

identifies $\left(\mathsf{Conf}_N(\mathbb{R}^2), \sum_{i=1}^N \Gamma_i \, \mathrm{pr}_i^* \, \omega\right)$ with a coadjoint orbit.

3. Vortex filaments

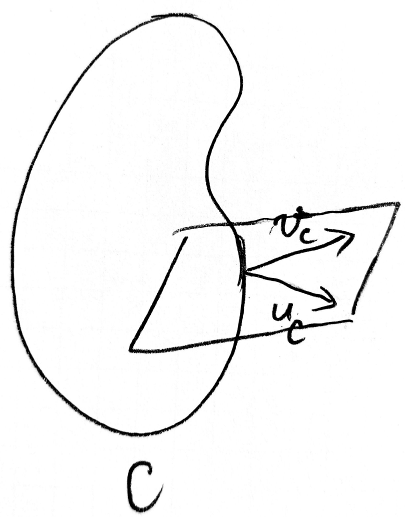
Space of closed curves

Let $G = \mathrm{Diff}_{\mathrm{vol},c}(\mathbb{R}^3)$ with

$$\mathfrak{g} = \mathfrak{X}_{\mathrm{vol},c}(\mathbb{R}^3) = \{X_\alpha : \alpha \in \Omega^1(\mathbb{R}^3),\ i_{X_\alpha}\mu = d\alpha\}.$$

G acts in a Hamiltonian way on the space of closed oriented unparametrized curves in $\mathbb{R}^3$, denoted by $\mathrm{Gr}_1(\mathbb{R}^3)$, endowed with the Marsden-Weinstein symplectic form:

$$\Omega_C(u_C, v_C) = \int_C i_{v_C} i_{u_C} \mu$$



Coadjoint orbit

The momentum map

$$J : \mathrm{Gr}_1(\mathbb{R}^3) \to \mathfrak{g}^*, \quad \langle J(C), X_\alpha \rangle = \int_C \alpha$$

identifies the vortex filaments space $\mathrm{Gr}_1(\mathbb{R}^3)$ with a coadjoint orbit.

Theorem [Haller-V.'04] Let $\dim M = n$ and $H^1(M) = 0$.
If $H^2(M) = 0$, all connected components of the codimension two Grassmannian $\mathrm{Gr}_{n-2}(M)$ are coadjoint orbits of $\mathrm{Diff}_{\mathrm{vol}}(M)$.
If $H^2(M) \neq 0$, the same holds for the Ismagilov central extension of $\mathrm{Diff}_{\mathrm{vol}}(M)$.

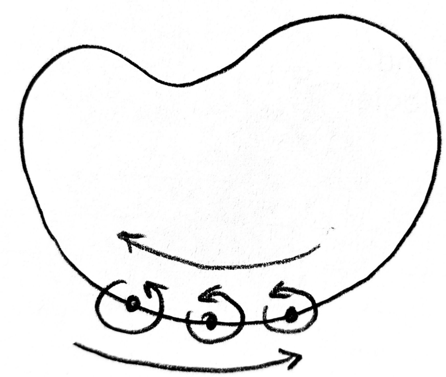
$$J : \mathrm{Gr}_{n-2}(M) \to \widehat{\mathfrak{g}}^*, \quad \langle J(N), [\alpha] \rangle = \int_N \alpha$$

$$[\alpha] \in \widehat{g} = \Omega^{n-2}(M)/d\Omega^{n-3}(M)$$

4. Vortex loops

Singular vorticity

We consider closed oriented curves C in the plane, endowed with a nowhere zero vorticity density $\beta \in \Omega^1(C)$.



The momentum $(C, \beta) \in \mathfrak{g}^*$ is

$$\langle (C, \beta), X_h \rangle = \int_C h\beta$$

with coadjoint action of $\varphi \in G$

$$\mathrm{Ad}^*_\varphi(C, \beta) = (\varphi(C), \varphi_* \beta)$$

Invariants

The only invariants of the coadjoint action are
- the enclosed volume $a = \int_C \nu$, where $\omega = d\nu$
- the total vorticity $w = \int_C \beta$

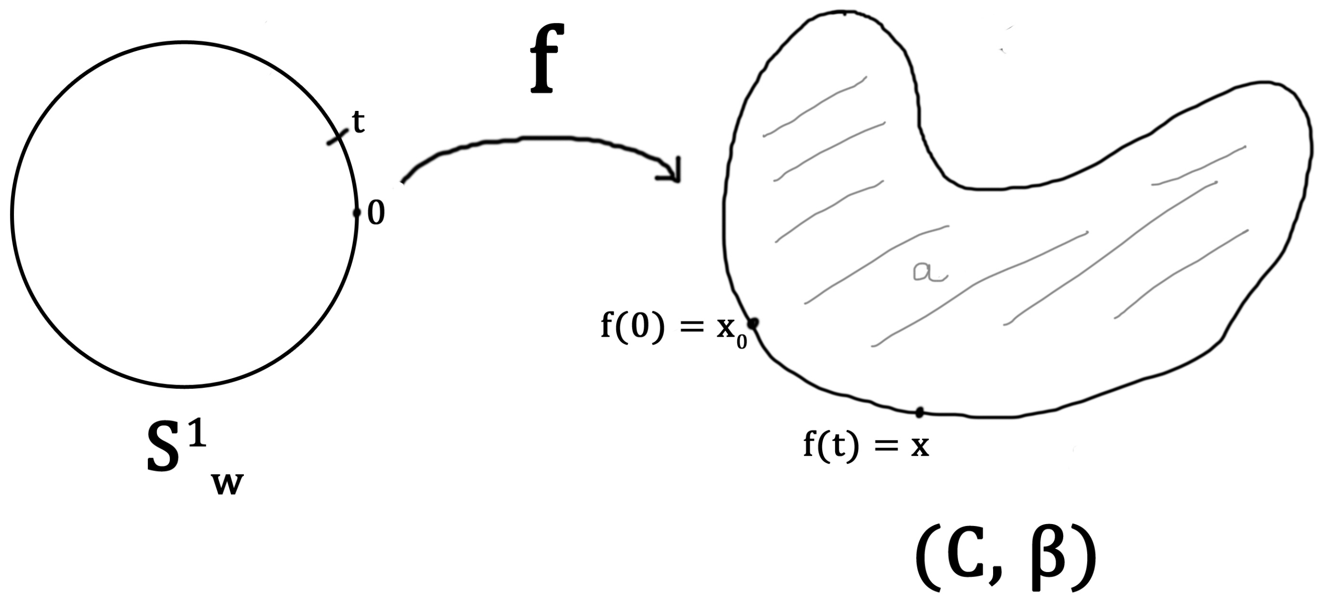
so the coadjoint orbits are

$$\mathcal{O}_{a,w} = \left\{ (C, \beta) : \int_C \nu = a \text{ and } \int_C \beta = w \right\}.$$

Each contains a circle with uniform distributed vorticity.

It is the 2D case of the coadjoint orbits considered in [Weinstein'90, Lee'09]: isodrastic leaves of the space of weighted Lagrangian submanifolds as coajoint orbits of the Hamiltonian group.

Parametrization modulo rotations



$$C = \operatorname{Im} f, \quad \beta = f_* \mathrm{d}t$$

Choosing a point $x_0 \in C$ with assigned parameter $t = 0$, an arbitrary point $x \in C$ has parameter $t = \int_{x_0}^{x} \beta$.

Coadjoint orbit of vortex loops

The space of votex loops becomes

$$\begin{aligned}\mathcal{O}_{a,w} &= \mathsf{Emb}_a(S^1_w, \mathbb{R}^2)/S^1_w \\ &= \{f : S^1_w \to \mathbb{R}^2 : \int_{S^1_w} f^*\nu = a\}/S^1_w\end{aligned}$$

Following our identification, we have

$$\langle [f], X_h \rangle = \int_0^w h(f(t))\mathrm{dt}$$

and the coadjoint action by pullback

$$\mathsf{Ad}^*_\varphi[f] = [\varphi \circ f]$$

KKS symplectic form

On $\mathrm{Emb}(S^1_w, \mathbb{R}^2)$ we have the natural symplectic form

$$\Omega_f(u_f, v_f) = \int_0^w \omega(u_f(t), v_f(t))\mathrm{dt}$$

The action of S^1_w by rotations is Hamiltonian with momentum map:

$$J : \mathrm{Emb}(S^1_w, \mathbb{R}^2) \to \mathbb{R}, \quad J(f) = \int_{S^1_w} f^*\nu$$

Symplectic reduction at $a > 0$ yields the symplectic manifold

$$\mathrm{Emb}_a(S^1_w, \mathbb{R}^2)/S^1_w = \mathcal{O}_{a,w}$$

with reduced symplectic form ω_{KKS}.

Canonical coordinates for ω_{KKS}

Using the Euclidean metric on the plane we have the decomposition

$$T_{(C,\beta)}\mathcal{O}_{a,w} = C_0^\infty(C) \times dC^\infty(C)$$

which corresponds to the decomposition of $T_f\,\mathrm{Emb}_a(S^1_w, \mathbb{R}^2)$ into normal and tangential coordinates:

$$u_f = \rho n + \lambda f', \quad \rho \in C_0^\infty(C), \lambda \in C^\infty(C).$$

The symplectic form becomes

$$\begin{aligned}(\omega_{KKS})_{(C,\beta)}((\rho_1, d\lambda_1), (\rho_2, d\lambda_2)) &= \int_C (\rho_1\lambda_2 - \rho_2\lambda_1)\mu_C \\ &= (\rho_1, d\lambda_2) - (\rho_2, d\lambda_1)\end{aligned}$$

for the non-degenerate pairing $(\rho, d\lambda) = \int_C \rho\lambda\mu_C$.

Prequantization

Theorem [Gay-Balmaz, V.] The coadjoint orbit $\mathcal{O}_{a,w}$ is prequantizable if and only if $aw \in 2\pi\mathbb{Z}$.

A prequantum bundle over a coadjoint orbit is a principal circle bundle $\mathcal{P} \to \mathcal{O}_\sigma$ endowed with a principal connection whose curvature is ω_{KKS}.
A character to $\sigma \in \mathfrak{g}^*$ is a group homomorphism $\chi : G_\sigma \to S^1$ that integrates $\sigma : \mathfrak{g}_\sigma \to \mathbb{R}$. (build $\mathcal{P} = G \times_{G_\sigma} S^1 \to G/G_\sigma = \mathcal{O}_\sigma$)

Theorem [Kostant, Tuynman] A character exists if and only if the coadjoint orbit is prequantizable and the coadjoint action can be lifted to an action on $\mathcal{P}$ by connection preserving automorphisms.

If $\pi_1(G) = \pi_2(G) = 0$, then $\pi_2(\mathcal{O}_\sigma) \cong \pi_1(G_\sigma)$.
(see [Neeb'02] for infinite dim.)

Natural 1-form

On the principal S^1_w-bundle $\pi : \mathrm{Emb}_a(S^1_w, \mathbb{R}^2) \to \mathcal{O}_{a,w}$ the 1-form

$$\Theta_f(v_f) = \int_0^w \nu(v_f(t))\mathrm{dt}$$

satisfies

$$d\Theta = \Omega = \pi^* \omega_{KKS},$$

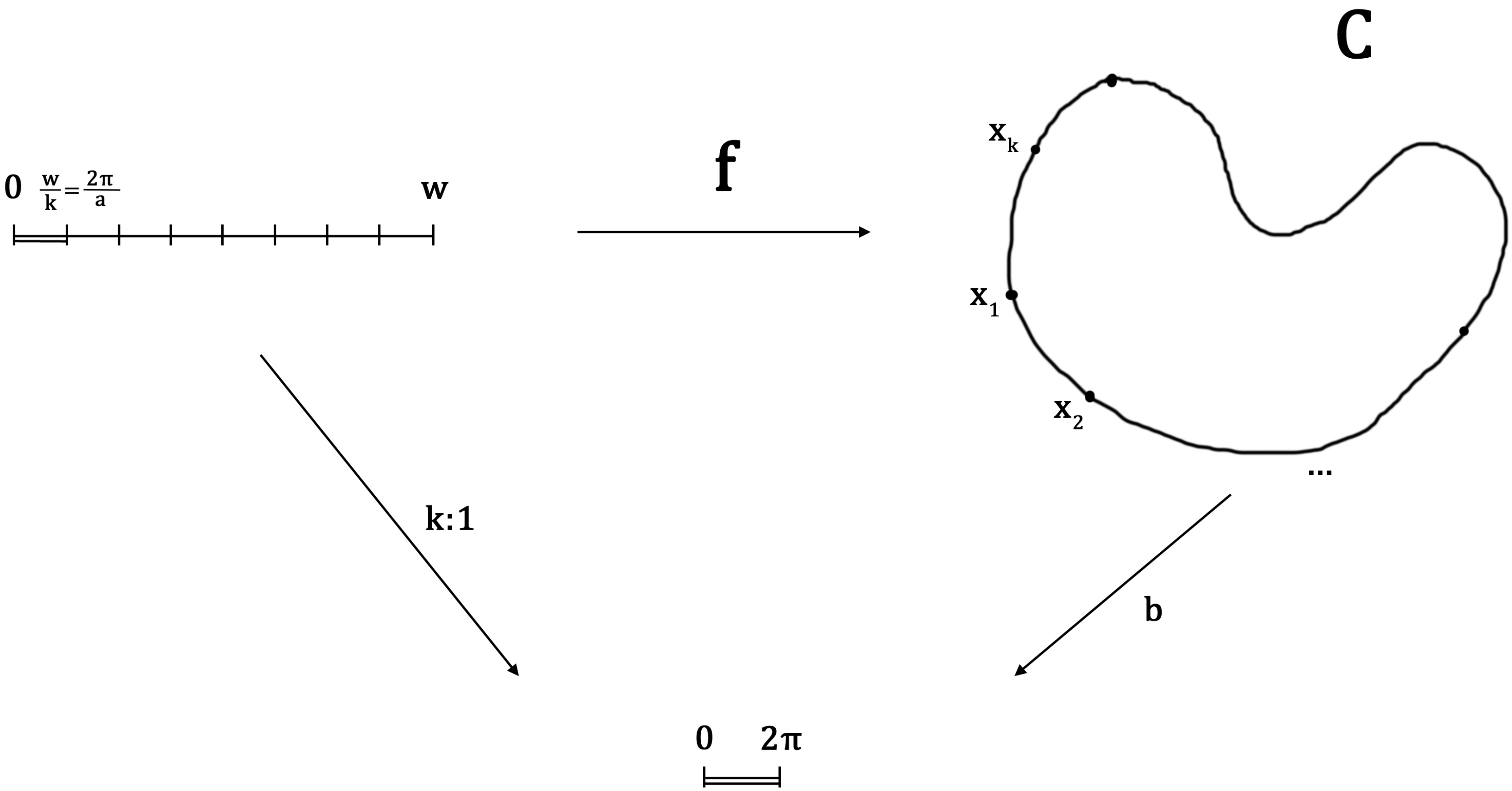
but Θ doesn't reproduces the infinitesimal generator of the circle action on the space of embeddings:

$$\Theta(f') = \int_C \nu = a,$$

so it is not a principal connection in general.

With the assumption $aw = 2\pi k$ with $k \in \mathbb{N}$, this can be remediated: we factorize by the subgroup $\mathbb{Z}_k \subset S^1_w$.

Factorization by $\mathbb{Z}_k$



$$f(\tfrac{w}{k}i) = x_{i+1} \text{ for } i = 0, 1, \ldots, k \text{ and } x_{k+1} = x_1$$
$$p^{-1}(0) = \{x_1, \ldots, x_k\}$$

Prequantum bundle

We factorize the space of embeddings through the action of $\mathbb{Z}_k$ by composition with rotations of integer multiples of $\frac{w}{k} = \frac{2\pi}{a}$:

$$\begin{aligned}\mathcal{P} &= \mathrm{Emb}_a(S^1_w, \mathbb{R}^2)/\mathbb{Z}_k \\ &= \{(C,p) \mid p : C \to S^1 \ k\text{-fold covering}\}\end{aligned}$$

with projection to $\mathcal{O}_{a,w}$ the logarithmic derivative divided by a:

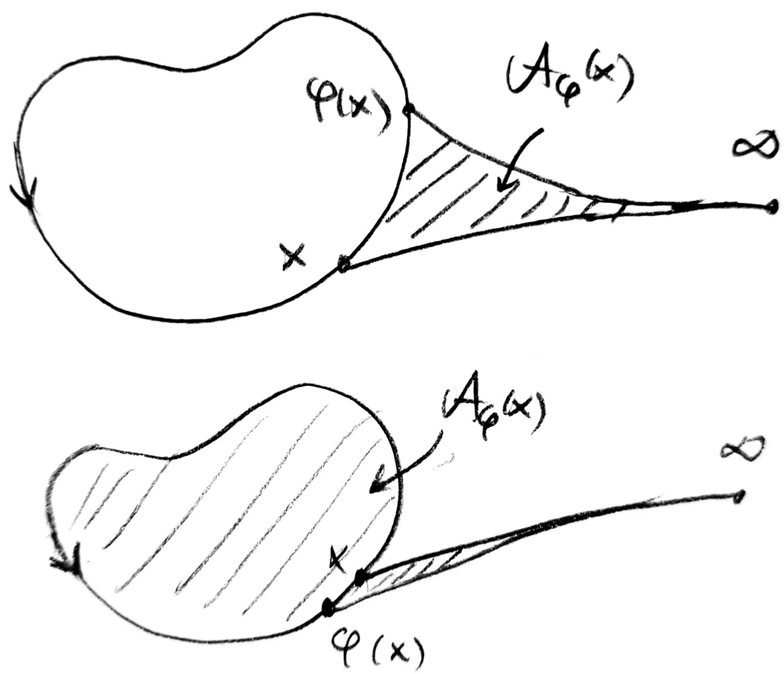
$$[f] = (C,p) \mapsto (C, \beta = \frac{1}{a} p^* dt).$$

The infinitesimal generator for the circle action on $\mathcal{P}$ is $\frac{1}{a} f'$, hence Θ descends to a principal connection 1-form on $\mathcal{P}$.

Character

The isotropy subgroup of (C, β) consists of Hamiltonian diffeomorphisms that preserve both C and β.

$$\chi(\varphi) = \int_C A_\varphi \beta \mod 2\pi\mathbb{Z}$$



Another formula for the character

Let $\nu \in \Omega^1(\mathbb{R}^2)$ such that $\omega = d\nu$.

For each Hamiltonian diffeomorphism $\varphi \in G$, there exists a unique $q_\varphi \in C_c^\infty(\mathbb{R}^2)$ that satisfies $\varphi^*\nu - \nu = dq_\varphi$.

Proposition [Gay-Balmaz, V.] Under the prequantization condition $aw \in 2\pi\mathbb{Z}$, the character is

$$\chi(\varphi) = \left(a \int_{x_0}^{\varphi^{-1}(x_0)} \beta + \int_C q_\varphi \beta \right) \mod 2\pi\mathbb{Z},$$

with $x_0 \in C$ arbitrary.

Polarization

As a polarization subgroup $H \subset G$ one takes the identity component of the Hamiltonian diffeomorphisms that preserve the curve C [Goldin, Menikoff, Sharp'87].

The character χ can be extended to H with a similar algebric formula

$$\chi(\varphi) = \Big(\int_C (\int_x^{\varphi^{-1}(x)} \beta) \ \nu(x) + \int_C q_\varphi \beta \Big) \mod 2\pi\mathbb{Z}.$$

The geometric formula holds on H too.

Pointed vortex loops

[Joint work with Ioana Ciuclea]
Let (C, β) be a vortex loop as before, assigned with $(x_1, \dots, x_k) \in \mathrm{Conf}_k(C)$ and the circulations $(\Gamma_1, \dots, \Gamma_k)$. We get a pointed vortex loop:

$$\langle (C, \beta, (x_i), (\Gamma_i)), X_h \rangle = \int_C h\beta + \sum_{i=1}^{k} \Gamma_i h(x_i)$$

The invariants of the (natural) coadjoint action:
- the enclosed volume a
- the total vorticity w
- $w_i = \int_{x_i}^{x_{i+1}} \beta$ with $w_1 + w_2 + ... + w_k = w$ (i.e. $\int_C \beta = w$)
- the circulations $\Gamma_1, ..., \Gamma_k$.

We identify yhe coadjoint orbit as $\mathrm{Emb}_a(S^1_w, \mathbb{R}^2)$ via

$$\Phi(f) = (f(S^1_w), f_* \mathrm{d}t, (f(w_1 + ... + w_i)), (\Gamma_i))$$

The KKS symplectic form

The KKS symplectic form on $\text{Emb}_a(S^1_w, \mathbb{R}^2)$ is

$$(\omega_{KKS})_f(u_f, v_f) = \int_0^w \omega(u_f(t), v_f(t))\text{dt} + \sum_{i=1}^k \omega(u_f, v_f)(\sum_{j=1}^i w_j)$$

is exact (hence prequantizable).

Using as before the decomposition into normal and tangent parts, the tangent space to the coadjoint orbit is

$$T_f \,\text{Emb}_a(S^1_w, \mathbb{R}^2) = C^\infty_0(C) \times C^\infty(C)$$

These are canonical coordinates for ω_{KKS} w.r.t. the non-degenerate pairing

$$(\rho, \lambda) = \int_C \rho\lambda\mu_C + \sum_{i=1}^k (\rho\lambda)(w_1 + \cdots + w_i)$$

5. Vortex sheets

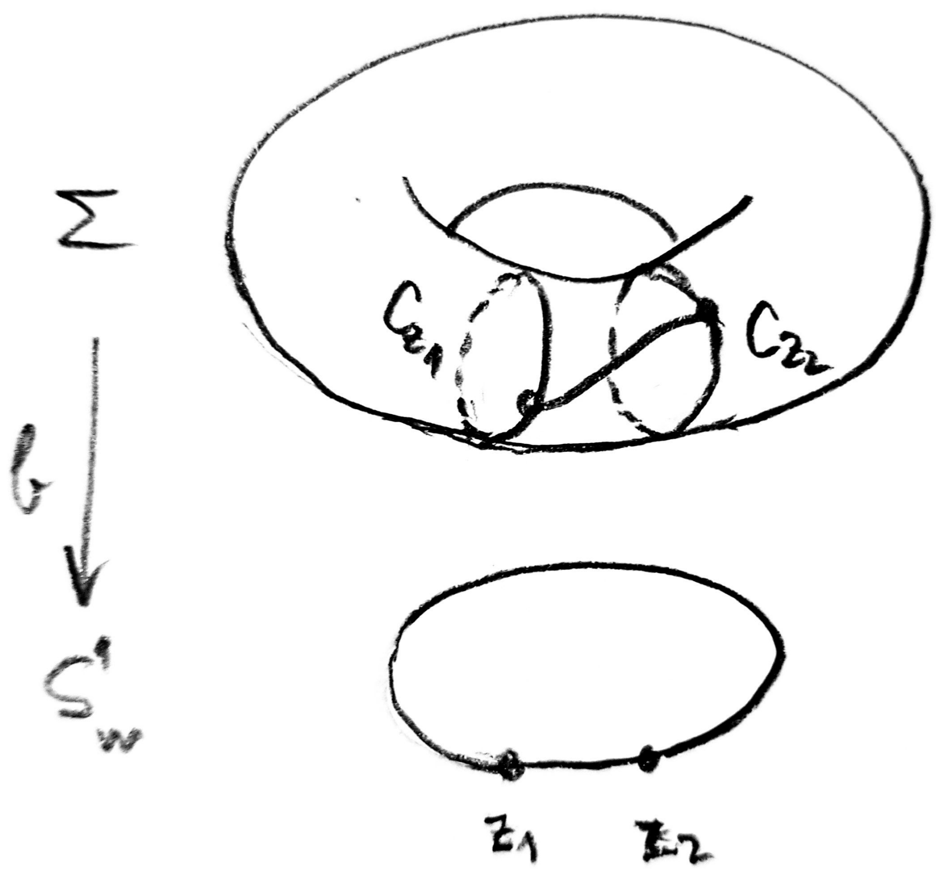
Vortex sheets in $\mathbb{R}^3$

Collection of vortex filaments, i.e. a surface Σ fibered in circles (filaments) with vorticity density $\beta \in \Omega^1(\Sigma)$ a closed 1-form such that the integrable distribution $\ker \beta$ is tangent to the fibers [Khesin'12].

Thus β nowhere zero (genus of Σ is 1) with discrete period group $w\mathbb{Z}$. There exist fibrations

$$b : \Sigma \to S^1_w$$

with logarithmic derivative β, i.e. $b^* dz = \beta$, up to multiplication by a constant element in S^1_w.



The vortex sheet as an element of $\mathfrak{g}^*$:

$$\langle(\Sigma,\beta),X_\alpha\rangle=\int_\Sigma\alpha\wedge\beta$$

Nonlinear Grassmannians

$$\mathrm{Gr}^S \quad \rightsquigarrow \quad \mathrm{Gr}^{S,\beta} \quad \rightsquigarrow \quad \mathrm{Gr}_a^{S,\beta}$$

$S \subset \mathbb{R}^3$ compact oriented surface of genus 1

$\mathrm{Diff}_+(S)$: group of orientation preserving diffeomorphisms of S.

$\mathrm{Emb}(S, \mathbb{R}^3)$: Fréchet manifold of embeddings of S into $\mathbb{R}^3$:

$$T_f \,\mathrm{Emb}(S, \mathbb{R}^3) = \left\{ u \circ f : S \to T\mathbb{R}^3 : u \in \mathfrak{X}(\mathbb{R}^3) \right\}.$$

$\mathrm{Gr}^S := \mathrm{Emb}(S, \mathbb{R}^3) / \mathrm{Diff}_+(S)$: nonlinear Grassmannian of all oriented surfaces in $\mathbb{R}^3$ of type S (including the orientations)

$$T_\Sigma \,\mathrm{Gr}^S = C^\infty(\Sigma)$$

$\rightsquigarrow$ Principal $\mathrm{Diff}_+(S)$-bundle

$$\pi : \mathrm{Emb}(S, \mathbb{R}^3) \to \mathrm{Gr}^S, \quad f \mapsto \Sigma = f(S),$$

$$T_f \pi : T_f \,\mathrm{Emb}(S, \mathbb{R}^3) \to T_\Sigma \,\mathrm{Gr}^S, \quad u \circ f \mapsto u|_\Sigma \cdot n,$$

Decorated nonlinear Grassmannians

$\mathrm{Diff}_+(S,\beta) \subset \mathrm{Diff}_+(S)$, where $\beta \in \Omega^1(S)$ nowhere zero.

$\mathrm{Gr}^{S,\beta} = \mathrm{Emb}(S,\mathbb{R}^3)/\,\mathrm{Diff}_+(S,\beta)$

With the ambient Euclidean metric we decompose the tangent space

$$T_{(\Sigma,\beta)}\,\mathrm{Gr}^{S,\beta} = C^\infty(\Sigma) \times dC^\infty(\Sigma).$$

$\rightsquigarrow$ Principal $\mathrm{Diff}_+(S,\beta)$-bundle

$$\pi^\beta : \mathrm{Emb}(S,\mathbb{R}^3) \to \mathrm{Gr}^{S,\beta}, \qquad f \mapsto (f(S), f_*\beta) = (\Sigma,\beta)$$

$$T_f\pi^\beta : T_f\,\mathrm{Emb}(S,\mathbb{R}^3) \to T_{\Sigma,\beta}\,\mathrm{Gr}^{S,\beta}, \qquad u \circ f \mapsto \left(u|_\Sigma \cdot n, \mathrm{d}(i_{u|_\Sigma^\top}\beta)\right)$$

The manifold of vortex sheets

μ Euclidean volume form on $\mathbb{R}^3$, $\nu \in \Omega^2(\mathbb{R}^3)$ such that $\mu = d\nu$.

Given $a > 0$:

$$\mathrm{Emb}_a(S, \mathbb{R}^3) := \left\{ f \in \mathrm{Emb}(S, \mathbb{R}^3) : \int_S f^* \nu = a \right\}$$

$$\mathrm{Gr}_a^S := \left\{ \Sigma \in \mathrm{Gr}^S : \int_\Sigma \nu = a \right\}$$

of oriented surfaces in $\mathbb{R}^3$ that enclose a constant volume a.

$$T_\Sigma \mathrm{Gr}_a^S \simeq C_0^\infty(\Sigma) := \left\{ f \in C^\infty(\Sigma) : \int_\Sigma f \mu_\Sigma = 0 \right\}, \quad \mu_\Sigma = i_n \mu.$$

$\mathrm{Diff}_+(S, \beta)$-bundle $\pi^\beta : \mathrm{Emb}_a(S, \mathbb{R}^3) \to \mathrm{Gr}_a^{S,\beta}$.

$$\mathrm{Gr}_a^{S,\beta} := \left\{ (\Sigma, \beta) \in \mathrm{Gr}^{S,\beta} : \Sigma \in \mathrm{Gr}_a^S \right\}$$

manifold of vortex sheets of type (S, β)

$$T_{(\Sigma,\beta)} \mathrm{Gr}_a^{S,\beta} = C_0^\infty(\Sigma) \times dC^\infty(\Sigma).$$

Symplectic form on vortex sheets

From $\beta \in \Omega^1(S)$ and volume $\mu \rightsquigarrow$ 2-form ω on $\mathrm{Emb}(S, \mathbb{R}^3)$:

$$\omega(u_1 \circ f, u_2 \circ f) = \int_S f^*(i_{u_2} i_{u_1} \mu) \wedge \beta, \quad u_1, u_2 \in \mathfrak{X}(\mathbb{R}^3).$$

Invariant under right $\mathrm{Diff}_+(S, \beta)$-action and left $\mathrm{Diff}_{\mathrm{vol}}(\mathbb{R}^3)$-action.

The restriction of ω to $\mathrm{Emb}_a(S, \mathbb{R}^3)$ descends via $\pi^\beta : \mathrm{Emb}_a(S, \mathbb{R}^3) \to \mathrm{Gr}_a^{S,\beta}$ to a symplectic form Ω on $\mathrm{Gr}_a^{S,\beta}$:

$$\omega = (\pi^\beta)^* \Omega.$$

In general Ω is not an exact form, even though its pullback ω is exact.

Canonical form for ω_{KKS}

Note that

$$T_{(\Sigma,\beta)}\,\mathrm{Gr}_a^{S,\beta} = T_\Sigma\,\mathrm{Gr}_a^S \times dC^\infty(\Sigma)$$

by using the nondegenerate pairing

$$C_0^\infty(\Sigma) \times dC^\infty(\Sigma) \to \mathbb{R}, \quad (\rho, d\lambda) = \int_\Sigma \rho\lambda\mu_\Sigma$$

The symplectic form Ω on $\mathrm{Gr}_a^{S,\beta}$ reads

$$\Omega_{(\Sigma,\beta_\Sigma)}\Big((\rho_1, d\lambda_1), (\rho_2, d\lambda_2)\Big) = -\int_\Sigma (\rho_1\lambda_2 - \rho_2\lambda_1)\mu_\Sigma.$$

(well defined since $\rho_1\mu_\Sigma$ and $\rho_2\mu_\Sigma$ exact forms on Σ)

Coadjoint orbit of vortex sheets

Left action of $\mathrm{Diff}_{\mathrm{vol}}(\mathbb{R}^3)$ on $\mathrm{Emb}_a(S, \mathbb{R}^3)$ descends to Hamiltonian action on $\mathrm{Gr}_a^{S,\beta}$:

$$\varphi \cdot (\Sigma, \beta_\Sigma) = (\varphi(\Sigma), \varphi_* \beta_\Sigma)$$

with equivariant momentum map

$$J : \mathrm{Gr}_a^{S,\beta} \to \mathfrak{X}_{\mathrm{vol}}(\mathbb{R}^3)^*, \quad \langle J(\Sigma, \beta_\Sigma), X_\alpha \rangle = \int_\Sigma \alpha \wedge \beta, \quad \alpha \in \Omega^1(\mathbb{R}^3).$$

- The restriction of the momentum map to any connected component of the manifold of vortex sheets $\mathrm{Gr}_a^{S,\beta}$ is one-to-one onto a coadjoint orbit

- The KKS symplectic form on the coadjoint orbit satisfies

$$J^* \omega_{\mathrm{KKS}} = \Omega.$$

Prequantization

Theorem [Gay-Balmaz, V.] The coadjoint orbit $\mathrm{Gr}^{S,\beta}$ is prequantizable if and only if $aw \in 2\pi\mathbb{Z}$.

A description of the prequantum bundle is by the space of possible fibration projections $b : \Sigma \to S^1_w$ (i.e. $\delta b = \beta$) factorized by the subgroup $\mathbb{Z}_k \subset S^1_w$.

Polarization subgroup $H \supset G_{(\Sigma,\beta)}$ consists of volume preserving diffeomorphisms that preserve the surface Σ [Goldin, Menikoff, Sharp'91]. They also obtain a prequantization condition of the same type, but for vortex ribbons/tubes.

Character

The Feynman-Onsager condition $aw \in 2\pi\mathbb{Z}$ also ensures the existence of a character

$$\chi : G_{(\Sigma,\beta)} \to S^1$$

as well as an extension to the polarization subgroup $H = G_\Sigma$ (actually the identity components of these groups).

Geometric description: let $\{\varphi_t\}$ be an isotopy from identity to φ and let D_z be any disk with boundary the fiber C_z. We denote by $V_\varphi(z)$ the volume swept out by D_z under the isotopy. Then

$$\chi(\varphi) = \int_{S^1_w} V_\varphi(z)dz \mod 2\pi\mathbb{Z}$$

Algebraic formula for the character

Given $\varphi \in G$, there exist 1-forms $q_\varphi \in \Omega^1(\mathbb{R}^3)$ such that

$$\varphi^*\nu - \nu = dq_\varphi$$

Proposition [Gay-Balmaz, V.] Under the prequantization condition $aw \in 2\pi\mathbb{Z}$, the character $\chi : H \to S^1$ is

$$\chi(\varphi) = \left(\int_\Sigma \Big(\int_x^{\varphi^{-1}(x)} \beta \Big) \nu(x) + \int_\Sigma q_\varphi \wedge \beta \right) \mod 2\pi\mathbb{Z}.$$

On the isotropy subgroup $\varphi \in G_{(\Sigma,\beta)}$ we have

$$\mathrm{Flux}_\beta(\varphi|_\Sigma) = \int_x^{\varphi^{-1}(x)} \beta \mod w\mathbb{Z}$$

where

$$\mathrm{Flux}_\beta : \mathrm{Diff}(\Sigma, \beta) \to S^1_w, \quad \mathrm{Flux}_\beta(\psi) = \frac{b \circ \psi^{-1}}{b}$$

References

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