Quantum decorated character stacks.

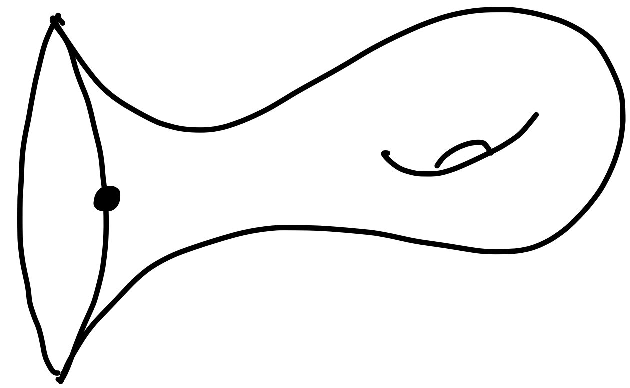
(joint work w/ D. Jordan, I. Le, G. Schrader)

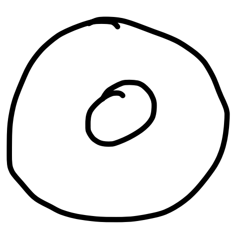
S – oriented surface

G – reductive group

$\underline{Loc}_{G,S} := \mathrm{Hom}(\pi_1(S), G)$

$Loc_{G,S} := \underline{Loc}_{G,S}/G$ – character "variety"

$S =$ 

<u>Ex:</u> $S =$ 

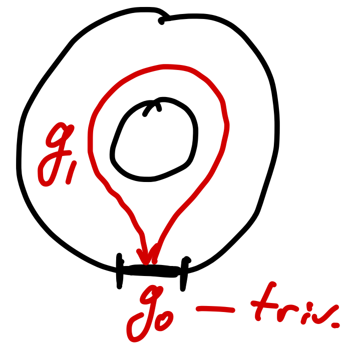
$Loc_{G,S} = G/G \leftarrow$ acts by conj.

$\Rightarrow Loc_{G,S}$ is <u>not</u> a variety

Fix: 1) Consider GIT quotient:

$$\mathrm{Loc}_{G,S} /\!\!/ G := \mathrm{Spec}\left(\mathcal{O}(\mathrm{Loc}_{G,S})^G\right)$$

2) Add trivialisation:

$S =$ 

$$\Rightarrow \mathrm{Loc}_{G,S} \simeq (G\times G)/G \simeq G$$

$$(g_0, g_1)h = (g_0 h, h^{-1} g_1 h)$$

[Atiyah, Bott] + [Goldman] + [Fock, Rosly]

$\mathrm{Loc}_{G,S}$ is a Poisson variety.

$\Rightarrow \mathrm{Loc}_{G,S}$ admits quantisation

Approaches to quantisation:

1) Via R-matrices [Fock, Rosly] + [Alekseev, Grosse, Schomerus]

2) Via skein algebras [Przytycki] + [Turaev]

3) Via cluster algebra [Fock, Goncharov] + [Goncharov, Shen]

4) Via factorisation homology [Ben-Zvi, Brochier, Jórdan]

<u>Goal</u>: relate 3) & 4).

<u>Remark</u>: 1), 2), 3) quantise $\mathcal{O}(\underline{Loc}_{G,S})^G$

$\Rightarrow$ only "see" affine variety

4) quantises the category of G-equiv. q.-coherent sheaves on $\mathcal{O}(\underline{Loc}_{G,S})$

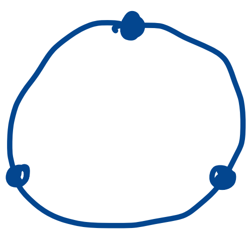
$\Rightarrow$ "sees" the whole stack

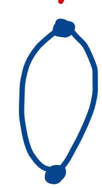
Cluster approach.

1) Instead of G-triv., consider sections of assoc. G/B- or G/N-bundles at marked pts $\{x_1,\dots,x_m\} \in \partial S$ or punctures

= bdry comp. w/o marked pts.

in generic pos-n

a)  $\mathcal{X}_{G,S} := (G/N)^3/G$

b)  $\mathcal{X}_{G,S} := (G/N)^2/G$

2) Δ-tr-n of S w/ vertices at marked pts or punctures. Glue $\mathcal{X}_{G,S}$ from $\mathcal{X}_{G,\Delta}$

Properties

1) $\mathcal{L}oc_{G,S}$ has cluster charts w/ $\{x_j, x_k\} = \varepsilon_{jk} x_j x_k$
log-canonical coord-s

2) $\Longrightarrow$ a canonical quantisation $\mathbb{L}^q_{G,S} = \mathcal{O}_q(\mathcal{L}oc_{G,S})$
along w/ a canonical Hilb. space rep-ns

$$\rho : \mathbb{L}^q_{G,S} \circlearrowright \mathcal{P}^{\lambda}_{G,S}$$

λ – eigenvalues of monodromies around punctures

3) ρ is Γ_S – equivariant

mapping class group of S

4) Conj: ρ respects cutting & gluing

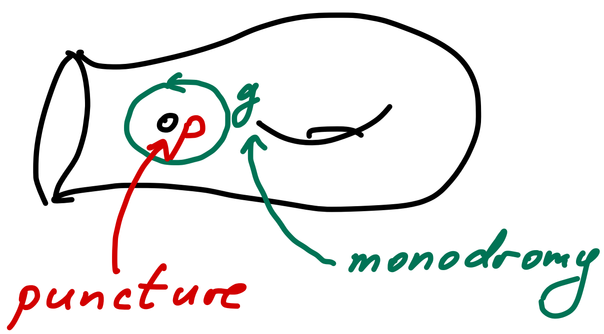
$\Longrightarrow$ yields a modular functor

Note: 1) Can't consider

$$S = \bigcirc \implies \mathrm{Loc}_{G,S} = pt/G$$

However $pt/G \rightsquigarrow$ G-equiv $QC(pt)$

$$\simeq \mathrm{Rep}(G)$$

2) 

$\implies \rho$ has a g-inv-t flag F: $gF = F$

a) $F \in G/B$

b) $F \in G/N \implies g$ – unipotent

Factorisation homology

Want: Fix G.

$$Z_q : \overset{\text{surface}}{S} \rightsquigarrow \overset{\text{category}}{Z_q(S)}$$

monoidal functor

"knows" about Γ_S-action

local in S

$\Rightarrow$ Γ_S-equivarience, modular functor

Thm: [Ben-Zvi, Francis, Nadler]

$Z(S) := \int_S \mathrm{Rep}(G)$ – factorisation homology w/ coeff-s in $\mathrm{Rep}(G)$

$\Rightarrow Z(S) \simeq QC(\mathrm{Loc}_{G,S})$ ⟵ G-equiv. quasi-coherent sheaves on $\underline{\mathrm{Loc}}_{G,S}$

[Ben-Zvi, Brochier, Jordan]

1) Quantisation $Z_q(S) := \int_S \underbrace{\mathrm{Rep}_q(G)}_{\text{integrable } U_q(\mathfrak{g})\text{-modules}}$

2) Relation to [Alekseev - Grosse - Schomerus]

Thm: [Jordan, Le, Schrader, S.]

$\mathbb{S}$ – stratified surface

$$Z_q(\mathbb{S}) := \int_{\mathbb{S}} Rep_q\left(G \leftarrow B \twoheadrightarrow T \right)$$

$\mathbb{S}$ – simple, $G = SL_2 / PGL_2$

$\Longrightarrow$ $\forall$ $\underline{\Delta}$ of $\mathbb{S}$ $\exists$ a "chart" $Z_q(\underline{\Delta})$

$$Z_q(\mathbb{S}) \supset Z_q(\underline{\Delta}) \simeq \mathcal{X}^q(\underline{\Delta}) - mod_{T_{\mathbb{S}}}$$

open subcategory

quantum torus

& $\mathcal{F}_{\underline{\Delta},\underline{\Delta}'} : Z_q(\underline{\Delta}) \longrightarrow Z_q(\underline{\Delta}')$ is given by cluster mut-s

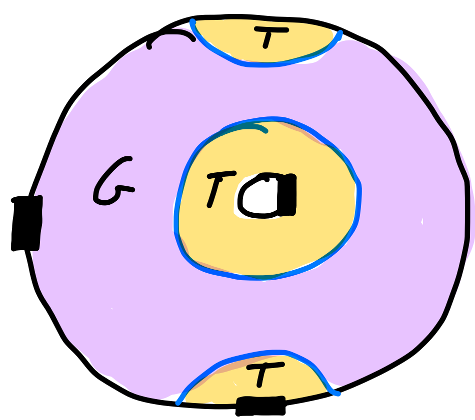
transition functors

Stratified surfaces.

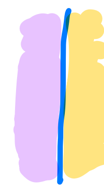
A **stratified** surface S is an oriented surface along with

1) collection of 1-dimensional "walls"
2) colouring of each 2-dim. region with G or T
3) collection of "gates"

Ex:



A stratified local system is a

1) G-local system on 
2) T-local system on
3) B-reduction of the $G\times T$-local system on 

   $B \hookrightarrow G$

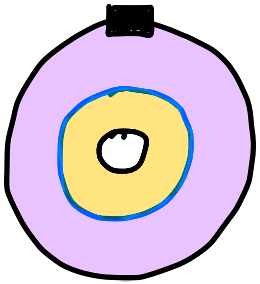
   $B \twoheadrightarrow T \simeq B/[B,B]$
4) Trivialisation at 

$Ch^{fr}_{G,S} :=$ stratified local system w/ trivialisation at gates

$Ch_{G,S} := Ch^{fr}_{G,S} / G_S \times T_S$

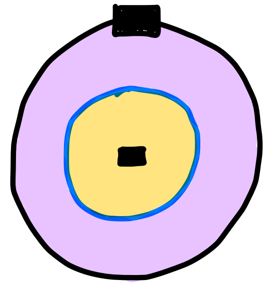
Ex:

$(G\times T)/B \simeq G/N$ $\quad$ G/B $\quad$ pt/N $\quad$ pt/B



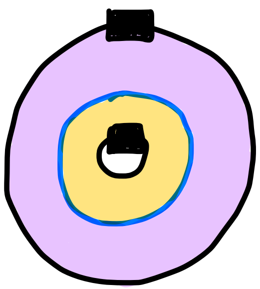
Springer resolution

$$\{(g, F) \mid gF = F\} \qquad g \in G, \quad F \in G/B$$



unipotent

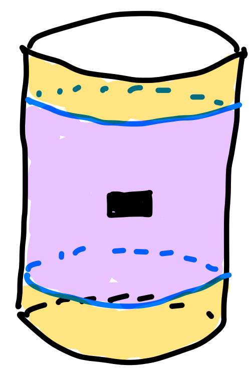
$$\{(g, \bar{F}) \mid g\bar{F} = \bar{F}\} \qquad g \in G, \quad \bar{F} \in G/N$$



$$\{(g, t, \bar{F}) \mid g\bar{F} = t^{-1}\bar{F}\}/G \qquad g \in G, \quad t \in T, \quad \bar{F} \in G/N$$

$$\simeq \{(g, \bar{F}) \mid g\pi(\bar{F}) = \pi(\bar{F})\}/G$$

$$\pi : G/N \longrightarrow G/B$$



multiplicative Steinberg variety

Stratified factorisation homology

[Ayala – Francis – Tanaka]

Consider sym. monoidal 2-category Surf:

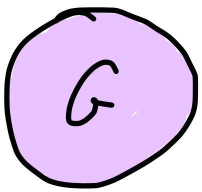
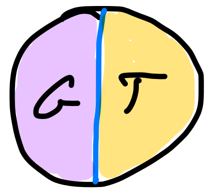
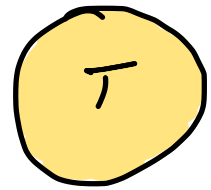
Obj = stratified surfaces

1-Mor = stratified embeddings

2-Mor = isotopies

$\otimes = \sqcup$

$\mathbb{D}isk \subset \mathcal{S}urf$ — the full subcategory generated by

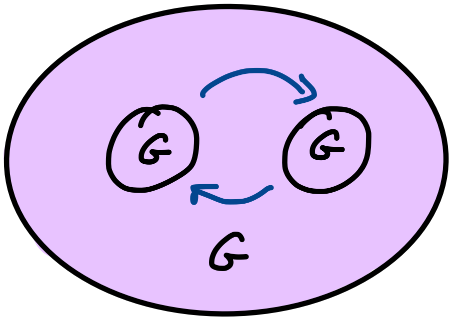
$\mathbb{D}_G =$ 
$\mathbb{D}_B =$ 
$\mathbb{D}_T =$ 

sym $\otimes$ functor

$$\begin{tikzcd}
\mathbb{D}isk \arrow[rr, "\mathcal{F}"] \arrow[dr, hook] & & \mathcal{C}at \\
& \mathcal{S}urf \arrow[ur, "\exists!\ \mathcal{Z}_q"'] &
\end{tikzcd}$$

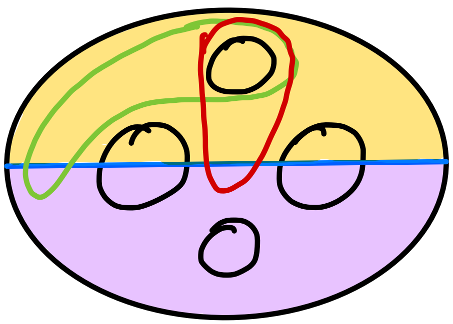
stratified factorisation homology

Structures on coefficients



$\Longrightarrow$ $\mathcal{F}(\mathbb{D}_G)$ — braided $\otimes$ category

$\mathcal{F}(\mathbb{D}_T)$ — — " —



$\Longrightarrow$ $\mathcal{F}(\mathbb{D}_B)$ — $\otimes$ category

$\mathcal{F}(\mathbb{D}_G) \boxtimes \mathcal{F}(\mathbb{D}_T) \longrightarrow \mathcal{F}(\mathbb{D}_B)$

$\otimes$ functor
w/ $\frac{1}{2}$-braiding

We set

$\mathcal{F}(\mathbb{D}_G) := \mathrm{Rep}_q(G)$ w/ R-matrix Braiding

$\mathcal{F}(\mathbb{D}_T) := \underset{\substack{\Downarrow \\ V}}{\mathrm{Rep}_q(T)}$ —— " ——, i.e.

V – spanned by weight vectors

$\sigma : v_\lambda \otimes v_\mu \longmapsto q^{(\lambda,\mu)} v_\mu \otimes v_\lambda$

$\mathcal{F}(\mathbb{D}_B) := \mathrm{Rep}_q(B)$

$$\begin{tikzcd}
\mathrm{Rep}_q(G) \arrow[dr, "i^*"'] & & \mathrm{Rep}_q(T) \arrow[dl, "\pi^*"] \\
& \mathrm{Rep}_q(B) &
\end{tikzcd}$$

$\frac{1}{2}$-Braiding: $V \in \mathrm{Rep}_q(G)$

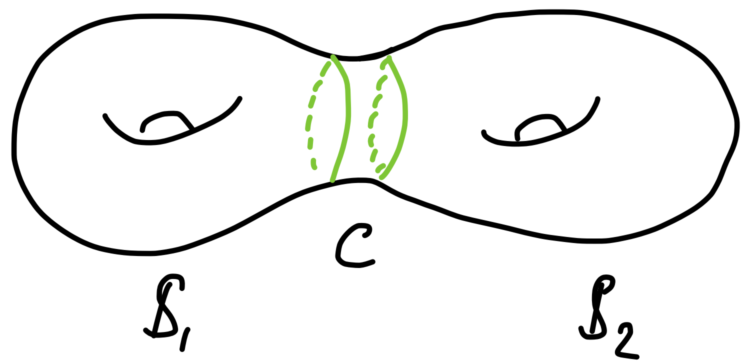
$$\Rightarrow \quad i^*(V) \otimes M \xrightarrow{\underset{\text{permutation}}{P} \otimes \underset{\in U_q(\mathfrak{b}_-) \otimes U_q(\mathfrak{b}_+)}{R}} M \otimes i^*(V)$$

Factorisation homology is

1) functorial:

$$\Sigma_1 \hookrightarrow \Sigma_2 \implies Z_q(\Sigma_1) \longrightarrow Z_q(\Sigma_2)$$

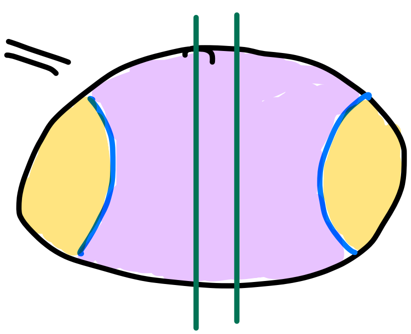
2) satisfies excision:



$Z_q(\Sigma_1)$, $Z_q(\Sigma_2)$ — module cat-s for $Z_q(C)$

& $Z_q(\Sigma) = Z_q(\Sigma_1) \boxtimes_{Z_q(C)} Z_q(\Sigma_2)$

$\mathbb{D}_2 =$



<u>Ex:</u> $Z_q(\mathbb{D}_2) = Z_q(\mathbb{D}_B) \boxtimes_{Z_q(\mathbb{D}_G)} Z_q(\mathbb{D}_B)$

$\mathrm{Rep}_q(B)$ as $\mathrm{Rep}_q(G)$-module

Classically, $\mathrm{Rep}(B) \xrightarrow{\sim} QC({}_G\backslash{}^G/_B)$
$M \longmapsto (G \times M)/_B$

$$QC({}_G\backslash{}^G/_B) \simeq QC({}_G\backslash{}^G/_N/_T)$$

G-equiv. sheaves on G/B $\qquad$ $(G\times T)$-equiv. sheaves on G/N

<u>Ex:</u> $G = SL_2$, $\quad G/N = \mathbb{C}^2 \setminus \{0,0\}$

$QC({}_G\backslash{}^G/_N/_T) \hookrightarrow \mathcal{O}(\mathbb{C}^2)-\mathrm{mod}_{G\times T}$

$QC(G/N) \simeq$ right orthogonal in $QC(\mathbb{C}^2)$ to sheaves, supported at 0

q-version:

Consider algebra $\mathcal{O}_q(G/N)$ in $\mathrm{Rep}_q(G\times T)$

$$\mathcal{O}_q(G/N) \simeq \bigoplus_{\lambda\in\Lambda^+} V_\lambda \otimes \chi_{-\lambda}$$

subcat. $\mathrm{Tors} \hookrightarrow \mathcal{O}_q(G/N)-\mathrm{mod}_{G\times T}$

torsion modules $\left(M\in \mathrm{Tors} \Leftrightarrow \forall m\in M \ \exists \mu\in\Lambda^+ \text{ s.t. } \forall \lambda \geq \mu \ \ V_\lambda\cdot m = 0\right)$

Thm: (Jordan – Le – Schrader – S.)

$$\mathrm{Rep}_q(B) \simeq \mathrm{Tors}^\perp \subset \mathcal{O}_q(G/N)-\mathrm{mod}_{G\times T}$$

$$\mathrm{Tors}^\perp = \{ \mathrm{Obj}: M\in \mathcal{O}_q(G/N)-\mathrm{mod}_{G\times T} \mid \forall N\in \mathrm{Tor} \ \ \mathrm{Hom}(N,M)=0\}$$

Computation of $Z_q(\mathbb{S})$

$$\mathcal{O}_q(G/N) \simeq \bigoplus_{\lambda \in \Lambda^+} V_\lambda \otimes \chi_{-\lambda}$$

Braided $\otimes$

$$\text{Schur's lemma} \Longrightarrow \left(\mathcal{O}_q(G/N) \,\tilde{\otimes}\, \mathcal{O}_q(G/N)\right)^{U_q(\mathfrak{g})} \simeq \mathbb{C}[\Delta_\lambda \mid \lambda \in \Lambda^+]$$

$$\begin{tikzcd}
Z_q(\mathbb{D}_2) \arrow[r, hook] & \mathcal{O}_q(G/N)^{\tilde{\otimes} 2}\text{-mod}_{G \times T^2} \\
\mathcal{O}_q(G/N)^{\tilde{\otimes} 2}[\Delta_\lambda^{-1}]\text{-mod}_{G \times T} \arrow[u] \arrow[ur] \arrow[d, "\wr\ \text{taking } G\text{-invts}"] & \\
\mathbb{C}[\Delta_\lambda^{\pm 1}]\text{-mod}_{T^2} &
\end{tikzcd}$$

act invertibly

conservative

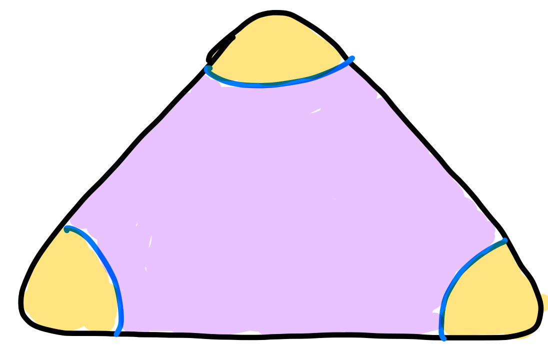
$\approx$

G acts freely on $(G/N)^2$ in generic pos-n

In a similar way

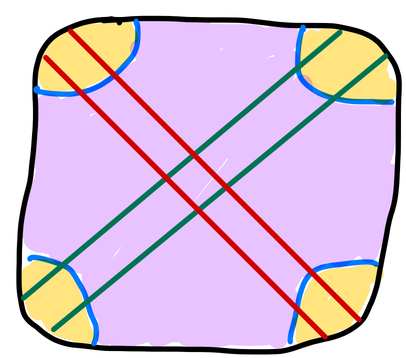
$$\mathcal{Z}_q(\mathbb{D}_3)$$
$$\cup$$
$$\mathbb{C}(q)[\Delta_{12}^{\pm 1}, \Delta_{23}^{\pm 1}, \Delta_{31}^{\pm 1}]\text{-mod}_T 3$$

$\uparrow$
quantum torus



$$\Delta_{12}\Delta_{23} = q\,\Delta_{23}\Delta_{13}$$

$$\mathcal{Z}_q(\mathbb{D}_4)$$
$$\cup \qquad \cup$$
$$\mathcal{Z}_q(\underline{\Delta}_{13}) \qquad \mathcal{Z}_q(\underline{\Delta}_{24})$$



$$\Delta_{13}\,\Delta_{24} = q^{-1}\Delta_{12}\,\Delta_{34} + q\,\Delta_{23}\,\Delta_{14}$$

Concluding remarks:

- quantises stacks
- a priori functorial construction
- open subcat. $Z_q(\Delta)$ allow to calculate
- can treat "G/N at punctures"
- can open more T-gates

  $\Rightarrow$ more elaborate "cluster" charts
- not applicable to cluster

  Hilbert space repn-s