Generalized geometry for three-manifolds

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choose coordinates (z_i)

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Complex m-manifold: choose coordinates (z_i)	On a real $2m$-manifold: consider locally (pointwise up to $\mathbb{C}^*$)
$\rho = dz_1 \wedge \ldots \wedge dz_m$ $\rho \wedge \overline{\rho} \sim$ volume $d\rho = 0$	$\rho \in \Omega^m_{\mathbb{C}}$ (dec.) $\rho \wedge \overline{\rho} \sim$ volume $d\rho = 0$
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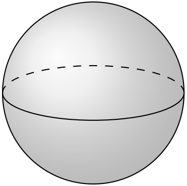
On $\mathrm{S}^2 = \mathbb{C} \cup \{\infty\}$,



dz on $\mathbb{C}$ and
$d(1/z)$ on $\mathbb{C}^* \cup \{\infty\}$
differ pointwise by $\mathbb{C}^*$
$d(1/z) = -dz/z^2$

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Symplectic structure
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$\operatorname{Ann} e^{i\omega} = \{0\}$
unsatisfactory

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$\mathrm{Ann}\, e^{i\omega} = gr(-i\omega)$
in $(TM \oplus T^*M)_{\mathbb{C}}$
for action
$(X + \alpha) \cdot \rho = \iota_X \rho + \alpha \wedge \rho$

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$\mathrm{Ann}\,\rho = T_{0,1} \oplus T^*_{1,0}$
(with $\rho = dz_1 \wedge \ldots \wedge dz_m$)
for the same action
$\longrightarrow$

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$T_{0,1} \oplus T^*_{1,0} \rightsquigarrow \mathrm{Ann}\,\rho \leftsquigarrow gr(i\omega)$

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$\updownarrow$ (pure pointwise)

$$\boxed{\begin{array}{c}\rho = e^{B+i\omega} \wedge \theta_1 \wedge \ldots \wedge \theta_r \\ \text{for } B, \omega \in \Omega^2,\ \theta_j \in \Omega^1_{\mathbb{C}}\end{array}}$$

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$(\rho,\overline{\rho}) := (\rho^T\wedge\overline{\rho})_{top}$

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$\mathrm{Ann}\,\rho$ involutive for Dorfman

$[X+\alpha, Y+\beta] = [X,Y] + L_X\beta - \iota_Y d\alpha$

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$\updownarrow$ (integrable)

$$\boxed{d\rho = v\cdot\rho \text{ for } v = X+\alpha}$$

Behind the scenes

$K = \langle \rho \rangle \subset \wedge^\bullet T^*_{\mathbb{C}} M$ line bundle

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$\wedge^\bullet T^*_{\mathbb{C}} M$ is a $\mathrm{Cl}_{\mathbb{C}}(TM \oplus T^*M)$-module

$\approx$ spinor representation,

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$$L := \mathrm{Ann}\, \rho$$

complex Dirac structure

(Courant algebroid $(TM \oplus T^*M)_{\mathbb{C}}$)

such that $L \cap \overline{L} = \{0\}$

References: Hitchin'03, Gualtieri'04/11, Alekseev-Bursztyn-Meinrenken'09

The type and a third example

Definition of **type**: r.

$\rho = dz_1 \wedge \ldots \wedge dz_m$ (type m)

$\rho = e^{i\omega} = 1 + i\omega + \ldots$ (type 0)

$\rho = e^{B+i\omega} \wedge \theta_1 \wedge \ldots \wedge \theta_r$

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$z = 0$, $\rho = dz \wedge dw$, pure of type 2

Some considerations

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  generically $\rho_0 \neq 0$ & when $\rho_0(p) = 0$, $d\rho_0(p) \neq 0$.

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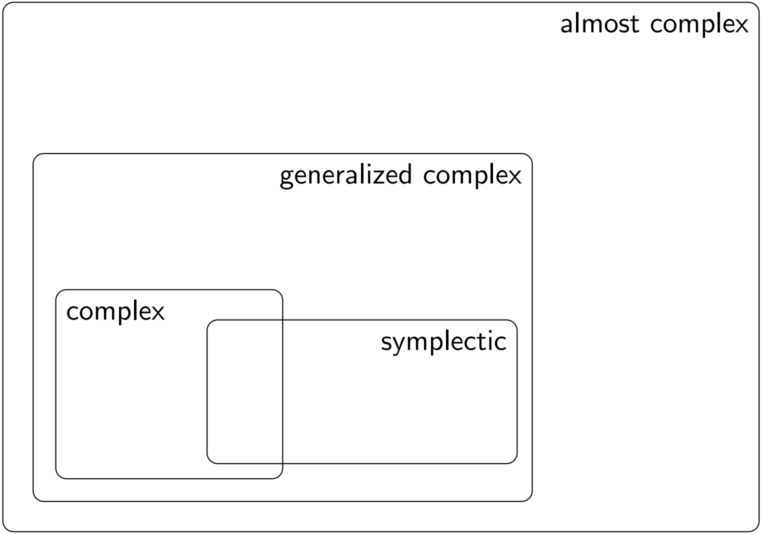
- Type-change only possible for $\dim M \geq 4$.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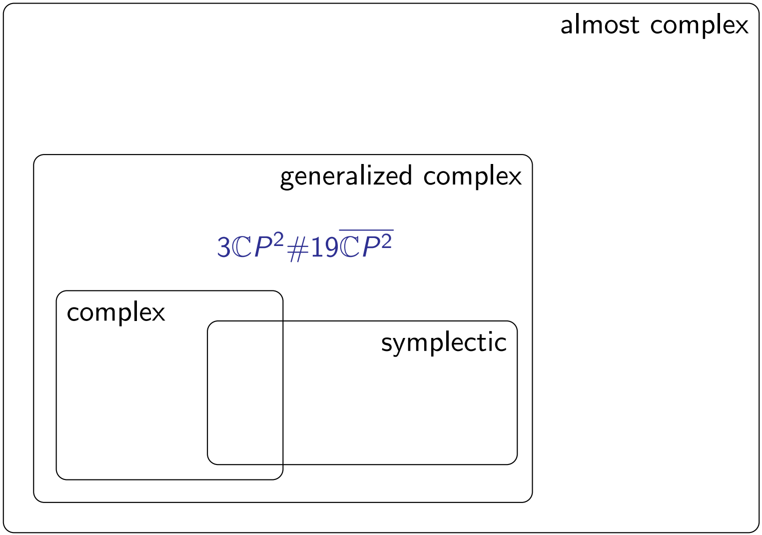
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Another generalized geometry is possible.


Nigel Hitchin - Generalized geometry of type B_n
GENERALIZED GEOMETRY
OF TYPE
Poisson 2012
Botón de reproducción (k)
0:04 / 58:49

- "extended" generalized geometry on M^n
- $T \oplus 1 \oplus T^*$
- inner product $(X + \lambda + \xi, X + \lambda + \xi) = i_X\xi + \lambda^2$
- $SO(n+1, n)$-structure – type B_n

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Example: **any usual generalized complex is B_n-generalized.**

Some considerations

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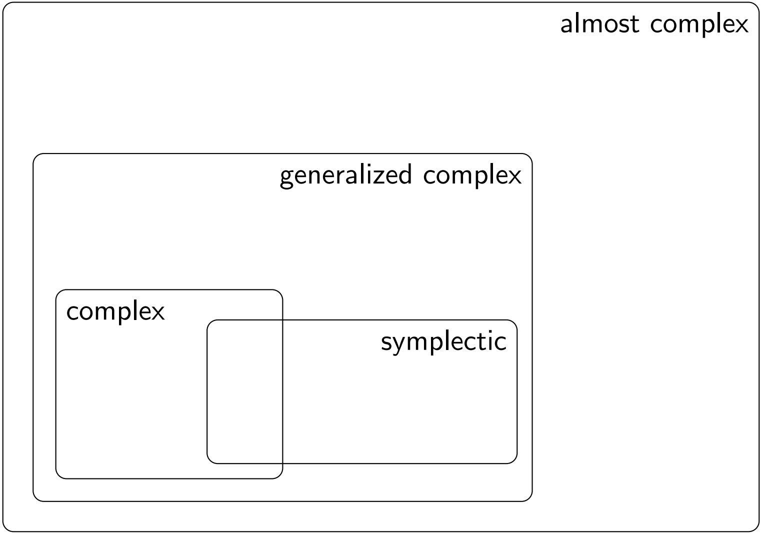

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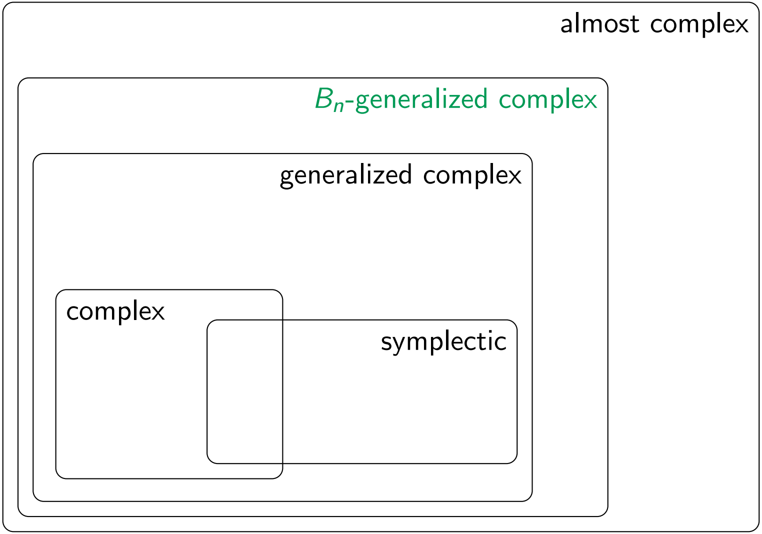

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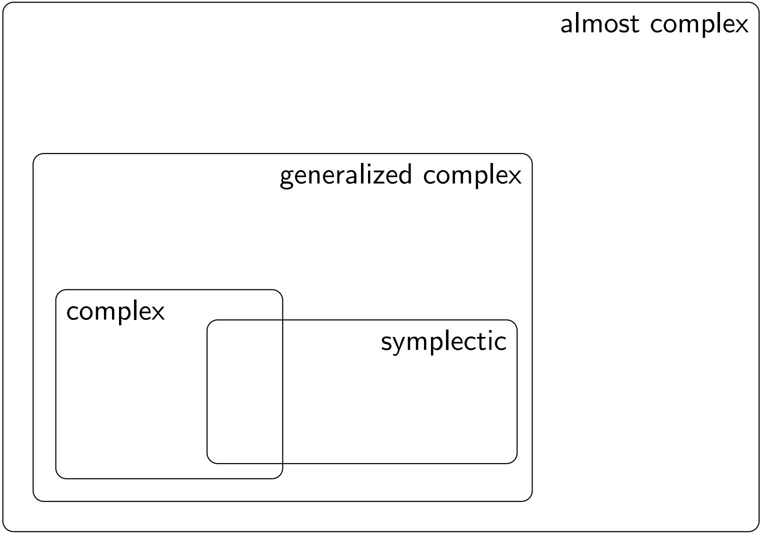

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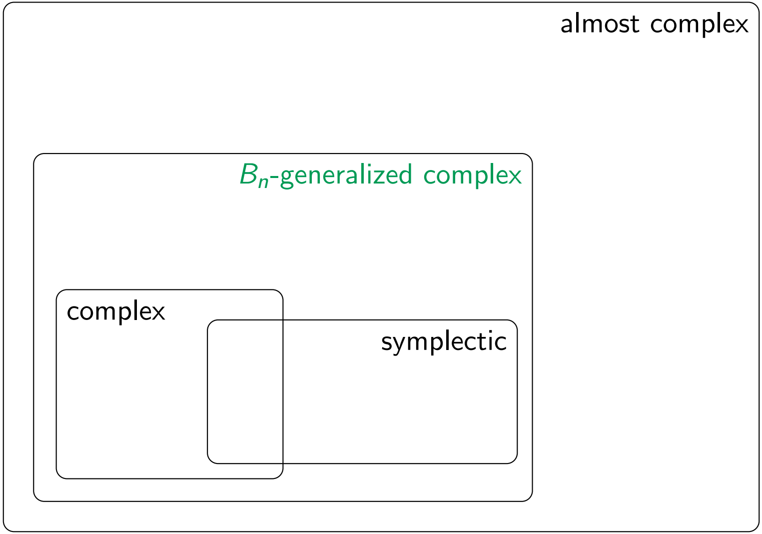

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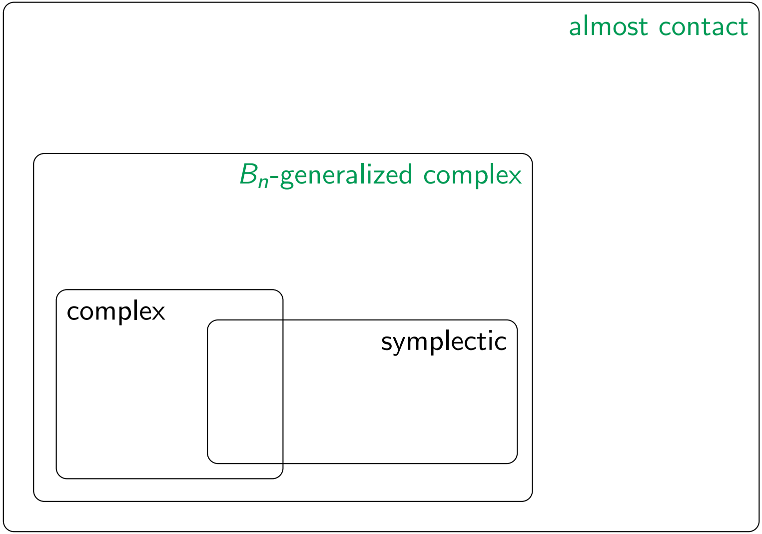

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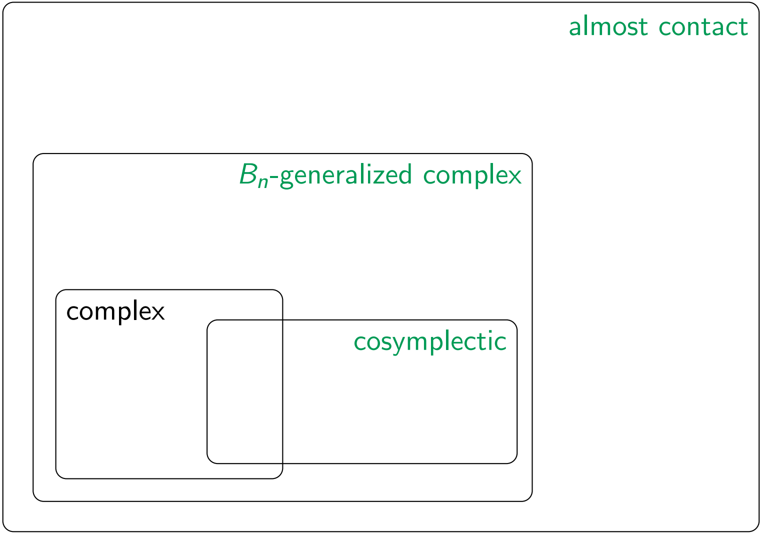

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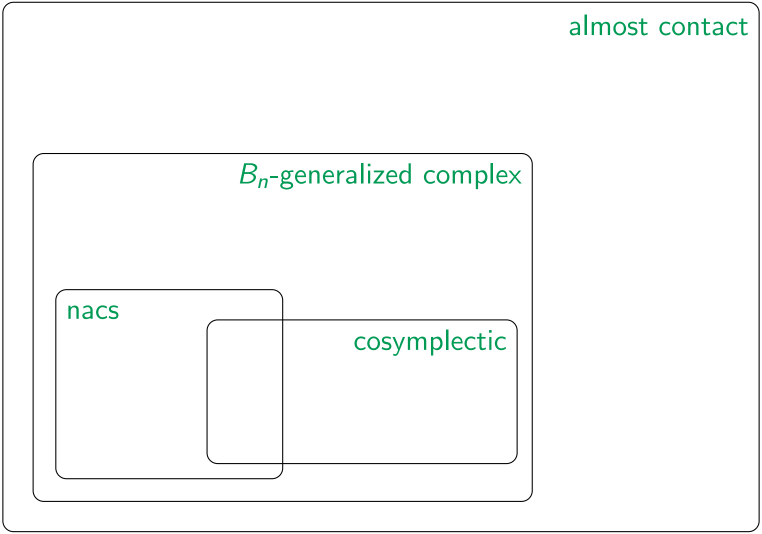

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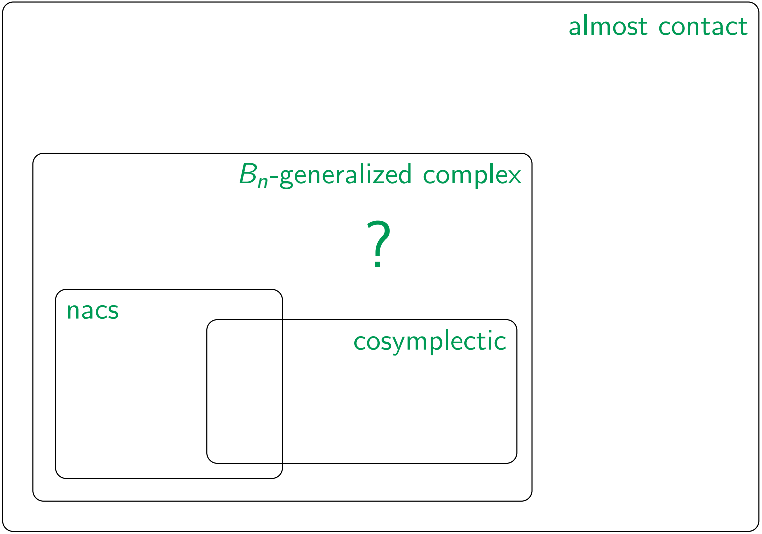

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We can do the same for 3-manifolds and $D^2 \times \mathrm{S}^1$!

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Act by a B-field $d\varphi \wedge dt$

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On the open manifold $M \setminus N_C$, Stokes' theorem says

$$\int_{\partial N_C} \iota^*(\rho_2/\rho_0) = \int_{M \setminus N_C} d(\rho_2/\rho_0) = \int_{M \setminus N_C} 0 = 0.$$

Contradiction. □

Example of type-change locus

Example (Hitchin)

Heuristically: $zw + dz \wedge dw$ on $\mathbb{C}^2 \setminus \{0\} \cong \mathrm{S}^3 \times \mathbb{R}^+$ should reduce to a

B_3-generalized complex structure on S^3
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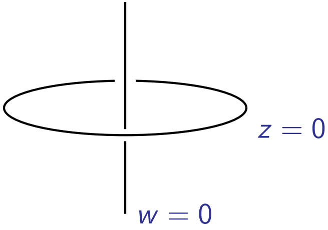

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Analogue of Marsden-Weinstein / Bursztyn-Cavalcanti-Gualtieri reduction:

Proposition (R.)

The reduction of an S^1 or $\mathbb{R}^+$-invariant generalized complex structure on $M \times \mathrm{S}^1$ or $M \times \mathbb{R}^+$ is a B_n-generalized complex structure on M.

The only type-change example has two circles.

I then realized:

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Time to move on:

The only type-change example has two circles.

I then realized:

Time to move on:
combine B_3 with geometrization

What geometric structures admit B_3-structures?

Joint work (in progress)
with **Joan Porti**

Marie Skłodowska-Curie
Individual Fellowship

GENERALIZED

Thurston's geometries

Thurston's geometries

Sol	H^3

$+$

Seifert	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	$S^2 \times \mathbb{R}$	E^3	$H^2 \times \mathbb{R}$
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Unlike for cosymplectic or normal almost contact...

Observation (Porti, R.)

For each Thurston geometry there is a geometric manifold admitting a B_3-generalized complex structure.

Thurston's geometries

Lemma (Porti, R.)

A geometric manifold that is neither cosymplectic nor normal almost contact has to be Sol or hyperbolic (not fibering over the circle), or the only euclidean manifold not fibering over the circle.

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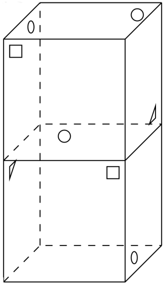

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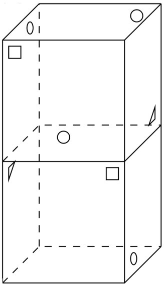

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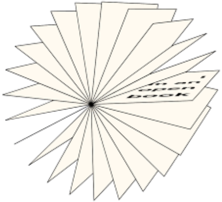

Open-book decomposition (general 3-fold)

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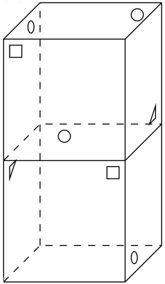

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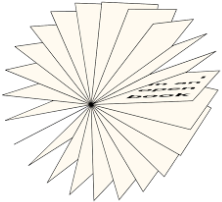

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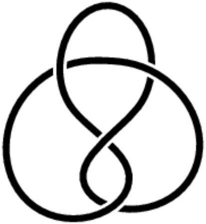

Complements of knots in $\mathbb{S}^3$ + surgery

Open-book decomposition

Defining B_3 on an open book:
- type-change disconnected binding,
- 'symplectic structure' on open leaves.

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1) unravel the S^3 structure:

Open-book decomposition

Defining B_3 on an open book:
- type-change disconnected binding,
- 'symplectic structure' on open leaves.

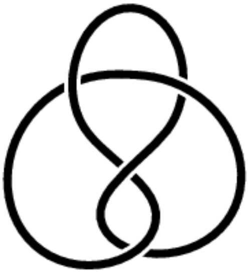
How to do it?
1) unravel the S^3 structure:
closed cylinder with Dehn twist,
2) modify the surface,
3) modify the twist

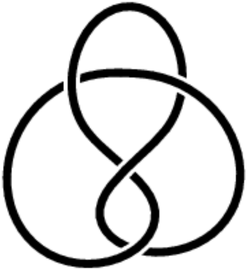
Open-book decomposition


MUSEO NACIONAL
DEL **PRADO**

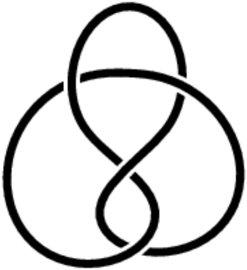
Open-book decomposition

MUSEO NACIONAL
DEL **PRADO**





But surgery does not work...



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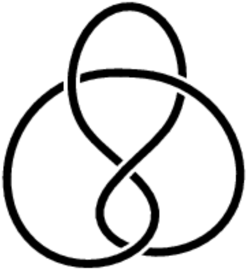

But surgery does not work...

I then realized:

Surgery works but changes the integrability of the structure.

Twisted generalized complex structures

Classical case:
for $H \in \Omega^3_{cl}$,

$\rho \in \Omega^\bullet_{\mathbb{C}}$ pure
$(\rho, \overline{\rho}) \sim$ volume.
$(d + H\wedge)\rho = v \cdot \rho$

Ševera class $[H] \in H^3$

Twisted generalized complex structures

Classical case:	B_n-case:
for $H \in \Omega^3_{cl}$,	for $F \in \Omega^2_{cl}$, $H \in \Omega^3$ such that $dH + F^2 = 0$
$\rho \in \Omega^\bullet_{\mathbb{C}}$ pure	$\rho \in \Omega^\bullet_{\mathbb{C}}$ pure
$(\rho, \overline{\rho}) \sim$ volume.	$(\rho, \overline{\rho}) \sim$ volume.
$(d + H\wedge)\rho = v \cdot \rho$	$(d + F \wedge \tau + H\wedge)\rho = v \cdot \rho$
Ševera class $[H] \in H^3$	A class $[(H, F)] \in$ " H_B^{3+2} "

(They correspond to Dirac structures on twisted Courant algebroids.)

Hinting at how the structure gets twisted

The two structures

$$\rho = 1 + idt + id\zeta \wedge d\psi - \ldots, \qquad \rho' = z + dz + idz \wedge dt$$

are combined via

$$\rho = e^{B_{ij} + A_{ij}\tau} \wedge f_{log}^{*}(\rho'/z)$$

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The integrability $(d\rho \dots)$ gets twisted by the dA_i and dB_i terms.
(this involves Çech trivializing $A_{ij} = A_i - A_j$, etc.)

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Impossible in an orientable 3-manifold!

In fact:

Proposition (Porti, R.)

A generalized surgery around a cosymplectic circle gives a type-changing structure twisted by exact F but the class $[(H, F)]$ is not trivial, that is, the structure is always twisted ($\rightarrow$ at least two circles as type change).

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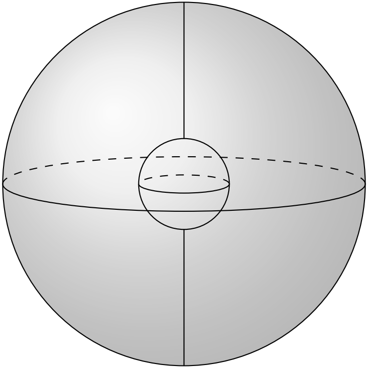
To get untwisted structures we would need at least two surgeries.

Simplest example



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$$\rho = 1 + idt + i\omega - dt \wedge \omega$$

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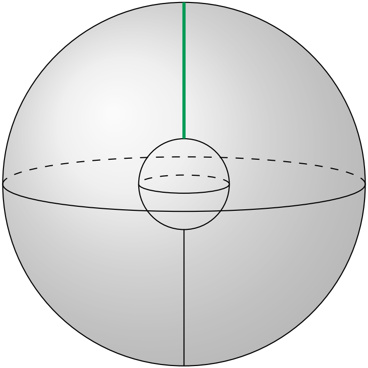

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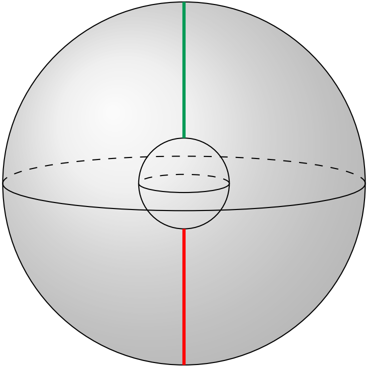

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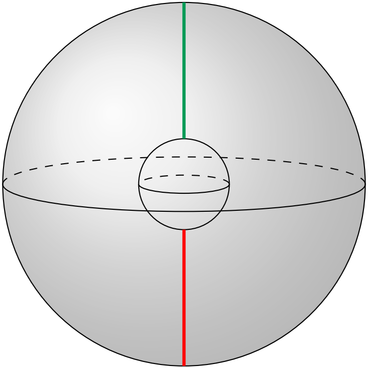

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And we are also changing the manifold!

Another surgery is possible

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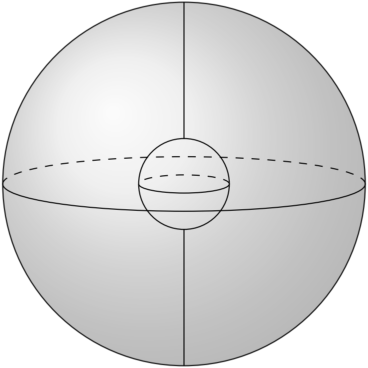

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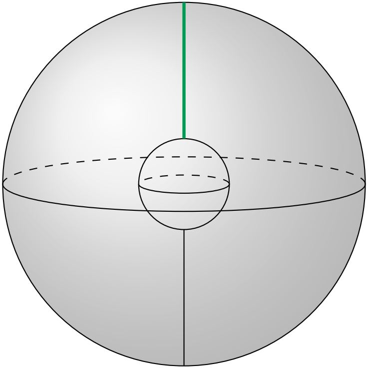

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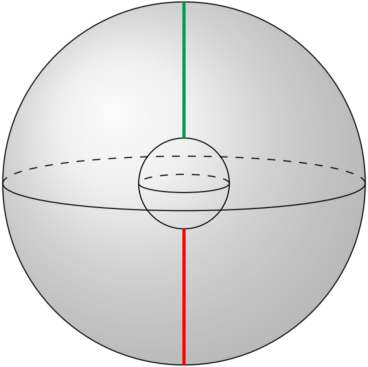

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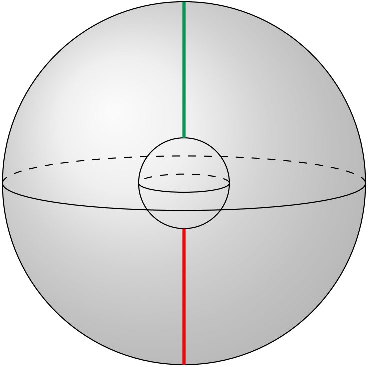

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But what is the manifold?

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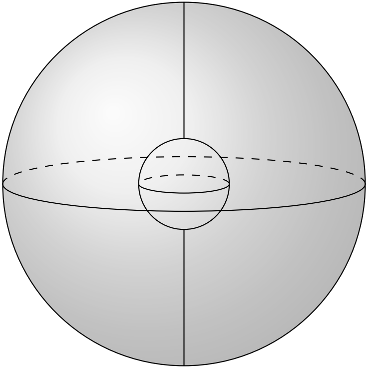
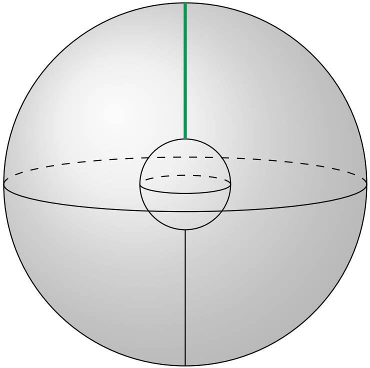


$$\mathrm{S}^2 \times \mathrm{S}^1$$

$$\mathrm{S}^2 = D^2 \cup_\partial D^2$$

Simplest example



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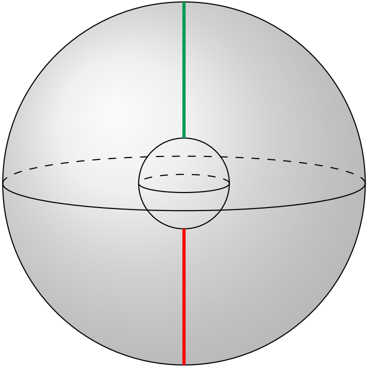

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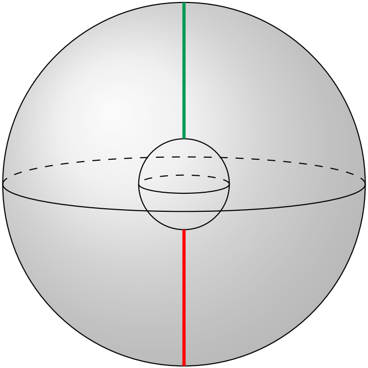

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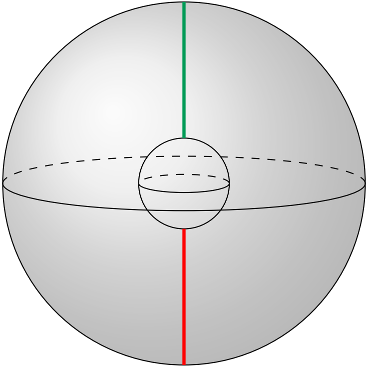

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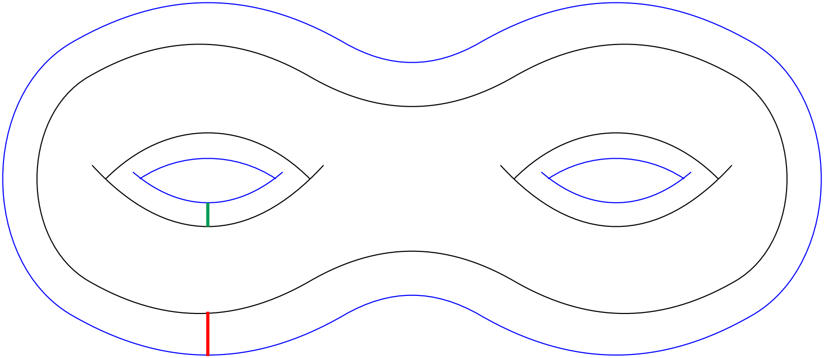
Same manifold... but what if we change S^2?

$\Sigma_g \times \mathrm{S}^1$



Perform two opposite surgeries around $P \times \mathrm{S}^1$ and $Q \times \mathrm{S}^1$:
B_3-generalized complex structure on...

What manifold do we get after two surgeries?

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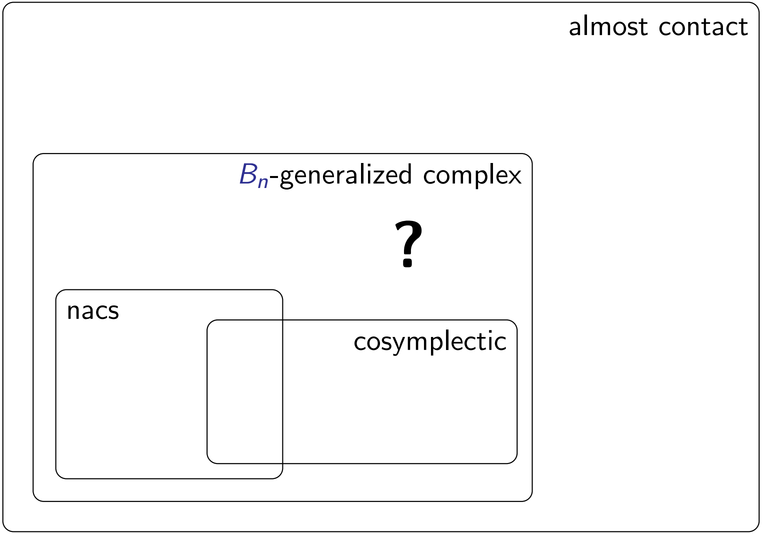
By repeating the surgeries (in pairs!)

Theorem (Porti, R.)

$\#^{2(g+s)-1}(\mathrm{S}^2 \times \mathrm{S}^1)$ admits a B_3-generalized complex structure whose type-change locus consists of $2s$ circles.


almost contact
B_n-generalized complex
?
nacs
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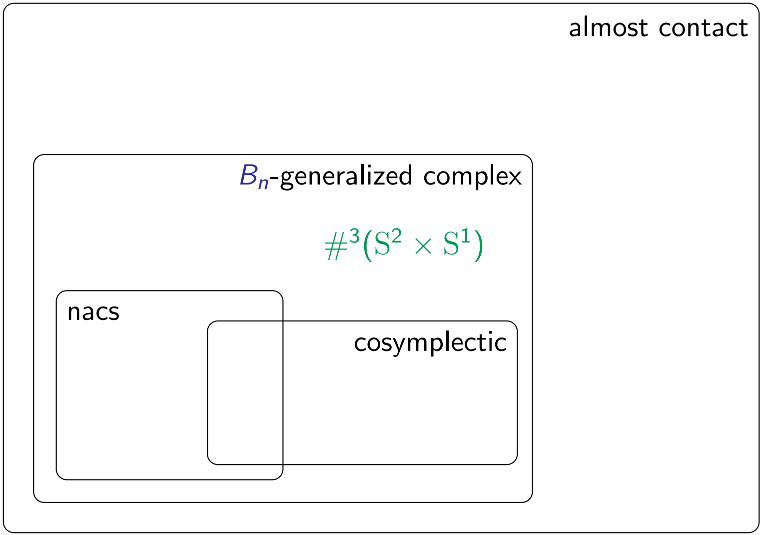
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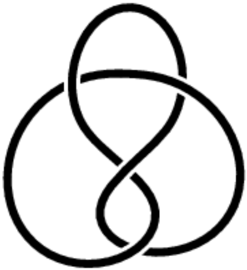
But it all started with surgeries for knots



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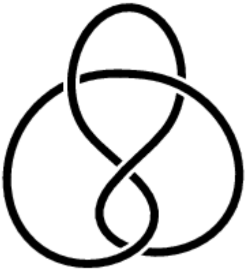

Theorem (Porti,R.)

There exist B_3-generalized complex structures on $N\#(\#^{2s-1}\mathrm{S}^2 \times \mathrm{S}^1)$, $s \geq 1$, where N is obtained from the mapping torus of a punctured surface by a Dehn filling along the S^1-direction.

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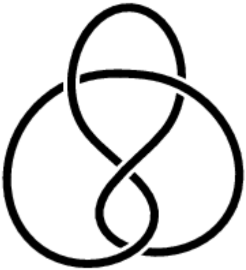

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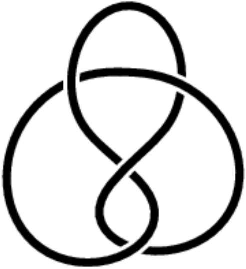

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- The type-change locus consists of $2s$ circles, $s \geq 1$.
- E.g., N any hyperbolic manifold coming from Dehn filling of $\mathrm{S}^3 \setminus K_8$.

Ideas of the proof

- Motivating example: $\mathrm{S}^3 \setminus K_8 \cong M_\psi$, mapping torus of

$$\psi := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}),$$ a diffeomorphism of $T^2 \setminus \{(0,0)\}$.

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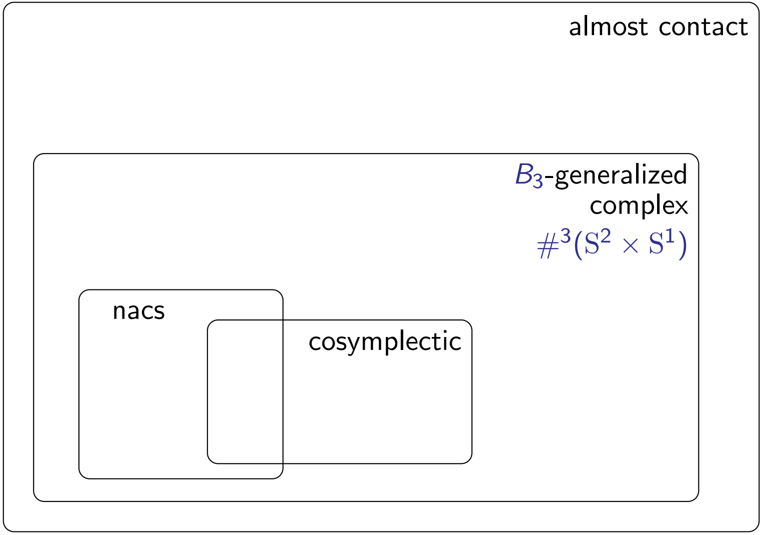
- Generalize to the mapping torus of a punctured surface by a Dehn filling along the S^1-direction.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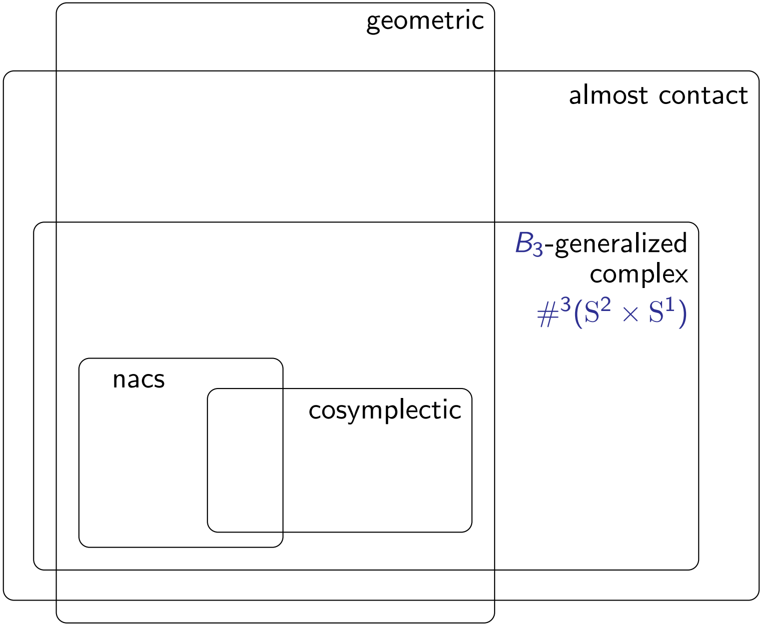
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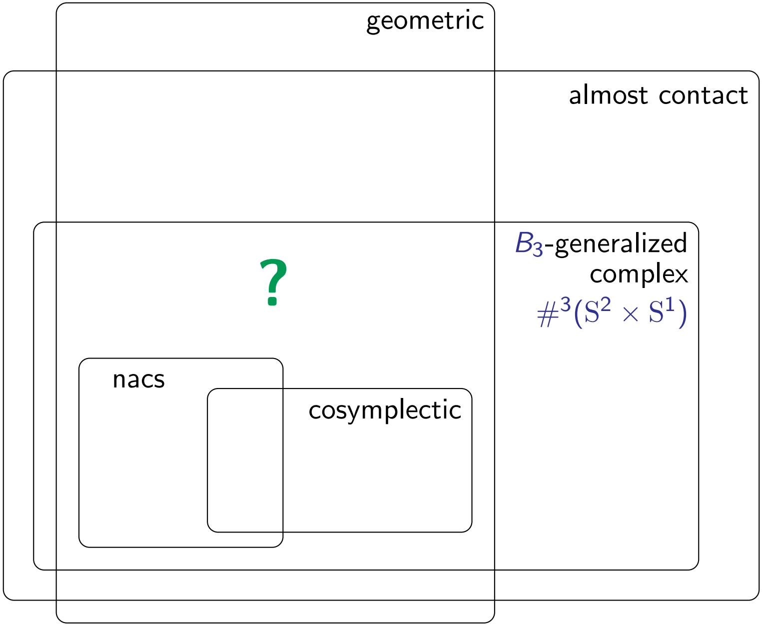
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$$2 \neq 1$$

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I will be in touch before Poisson 2032.

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Thank you for your attention!

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