Haefliger structures and wrinkling

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Before we start

A better title for the talk would have been “Foliations endowed with a transverse geometry”.

The talk will deal with a general notion of “geometry” and will tackle topological aspects of foliations.

The starting point of our project (apart from the pioneering work of Haefliger) was the paper

- F. Laudenbach, G. Meigniez. *Haefliger structures and symplectic/contact structures*,

that deals with the contact and symplectic cases.

Hopefully motivating examples

Baby example

Given a manifold M, consider the space of functions $C^\infty(M)$.

- $C^\infty(M)$ is contractible. Classification up to homotopy is uninteresting.
- Many classes up to diffeomorphism/isotopy.

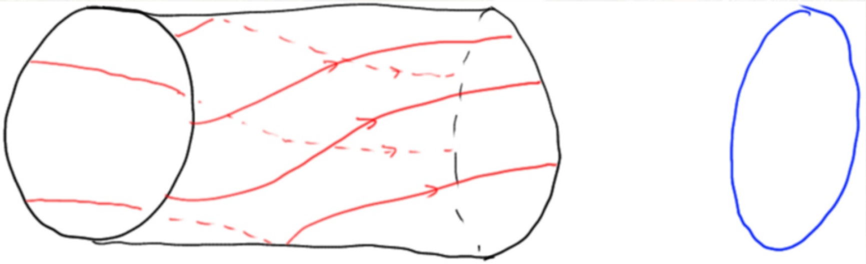
$f_0, f_1 \in C^\infty(M)$ are **concordant** if there are:

- A function $h : M \times [0,1]$ restricting to $M \times \{i\}$ as f_i,
- A 1-dim foliation $\mathcal{F}$ on $M \times [0,1]$ transverse to $M \times \{0,1\}$,
- such that h is constant along the leaves of $\mathcal{F}$.

Question: What is $C^\infty(M)/_{\text{concordance}}$?

Baby example

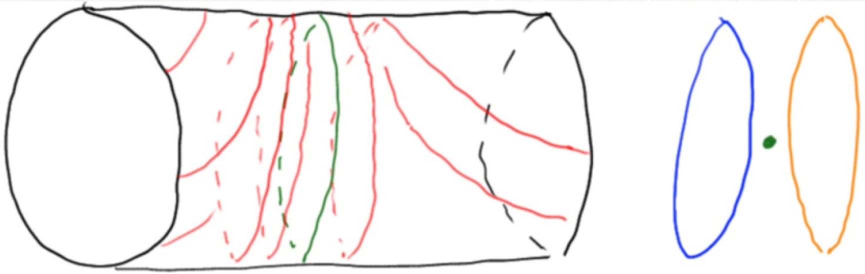
Set $M = \mathbb{S}^1$. Foliations on $\mathbb{S}^1 \times [0,1]$?



First case: *Suspension* of diffeomorphism φ.
Fact: All other foliations (up to diffeo rel. boundary) have a circular leaf.

Baby example

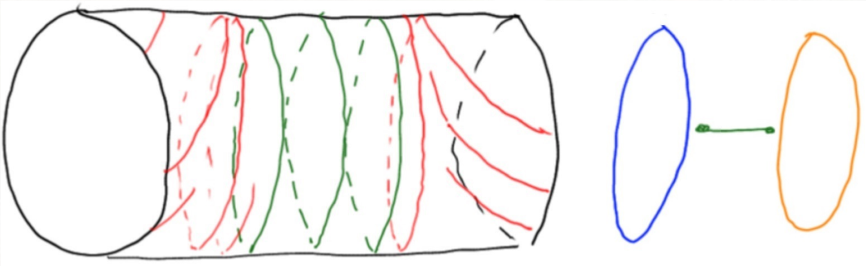
Set $M = \mathbb{S}^1$. Foliations on $\mathbb{S}^1 \times [0,1]$?



Second case: There is a single circular leaf.

Baby example

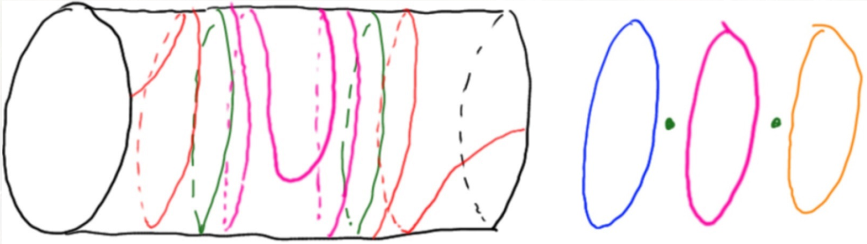
Set $M = \mathbb{S}^1$. Foliations on $\mathbb{S}^1 \times [0,1]$?



Third case: An interval of circular leaves.
Modification: We can add holonomy in the $\mathbb{S}^1$-direction using a diffeo of the interval.

Baby example

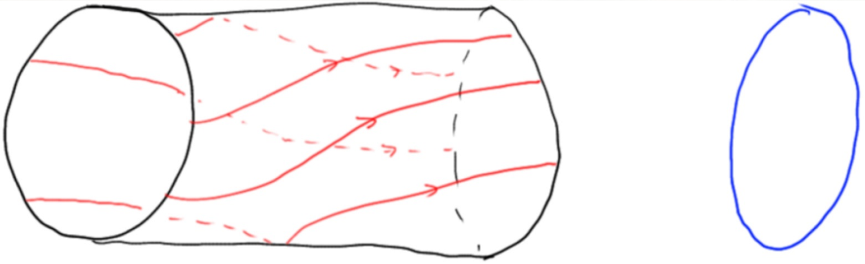
Set $M = \mathbb{S}^1$. Foliations on $\mathbb{S}^1 \times [0,1]$?



Fourth case: Two circular leaves bounding a Reeb component.
Fact: All foliations in the cylinder can be obtained as a concatenation of these.

Baby example

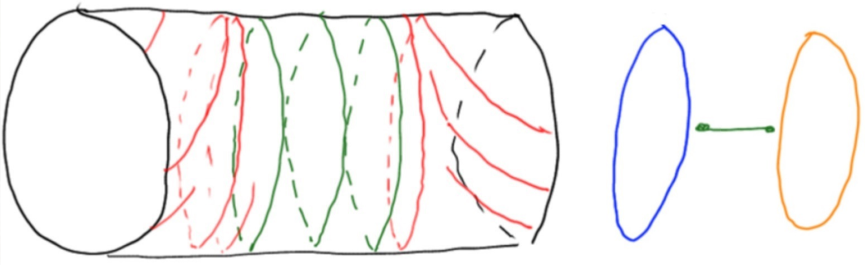
Question: What is $C^\infty(\mathbb{S}^1)/_{\text{concordance}}$?



Suppose f_0 and f_1 are concordant through a suspension. Then, $f_0 = f_1 \circ \varphi$.

Baby example

Question: What is $C^\infty(\mathbb{S}^1)/_{\text{concordance}}$?



Suppose they are concordant by a foliation with a circular leaf. Then, f_0 and f_1 are constant (but not necessarily the same!).

Baby example

Question: What is $C^\infty(\mathbb{S}^1)/_{\text{concordance}}$?

Answer: $C^\infty(\mathbb{S}^1)/_{\text{concordance}}$ is

$$\left(C^\infty(\mathbb{S}^1) \setminus \{\text{constants}\}\right) /_{\mathrm{Diff}_0(\mathbb{S}^1)} \quad \cup \quad \{[\text{constants}]\}.$$

Punchline: Close to classification up to isotopy, not homotopy.

Less trivial example

Let M be closed, $2n$-dimensional. Write

$$\mathrm{Symp}(M) := \{\omega \in \Omega^2(M) \mid \omega^n \neq 0, d\omega = 0\}.$$

Classification up to homotopy (with fixed cohomology class) agrees with isotopy (Moser stability). Classification up to isotopy/homotopy is a central problem in Symplectic Topology.

$\omega_0, \omega_1 \in \mathrm{Symp}(M)$ are **concordant** if there is:

- A closed 2-form β with $\beta^n \neq 0$ on $M \times [0,1]$,
- restricting to $M \times \{i\}$ as ω_i.

That is, the kernel of β is a rank-1 transversely symplectic foliation.

Less trivial example

Theorem (McDuff)

Two symplectic forms ω_0 and ω_1 are concordant if and only if they are cohomologous and homotopic (stably) as maximal-rank 2-forms.

In particular, concordant if they have the same formal data.

McDuff also proved the analogous result for contact.

I.e. concordance destroys all subtle symplectic/contact invariants. In particular, a tight and a overtwisted contact structures may be concordant.

Punchline: For contact/symplectic, concordance is more flexible than homotopy!

But why?

Contact and symplectic structures are special:

- They have a local model.
- They are globally stable (in closed manifolds).
- They have a large automorphism group.
- Which is n-transitive.
- Which satisfies fragmentation (in Hamiltonian case).
- They extend to flexible geometries in $+1$-dimension.

We expect “less nice” geometries to be rather rigid up to concordance.

Transverse geometries

Geometries

Consider the case of symplectic structures:

- they are sections of $\wedge^2 T^*M$,
- satisfying the differential conditions $\omega^n \neq 0$, $d\omega = 0$,
- which are invariant under diffeomorphisms,
- they form a sheaf $\mathrm{Op}(M) \to$ Top,
- also Diff-invariant.

Geometries

Similarly, we want:

- A bundle $X \to M$ with a $\mathrm{Diff}(M)$-action,
- a Diff-invariant differential condition $\mathcal{R}$ for sections of X.

So we consider the Diff-invariant sheaf

$$\mathcal{G} : \mathrm{Op}(M) \longrightarrow \text{Top}$$

of sections of X that satisfy $\mathcal{R}$.

This is induced from a sheaf

$$\mathcal{G} : \mathrm{Man}_q \longrightarrow \text{Top}.$$

Transverse geometries

A **transverse $\mathcal{G}$-geometry** on $(M, \mathcal{F})$ is a solution σ of $\mathcal{G}$ on the leaf space $M/\mathcal{F}$.

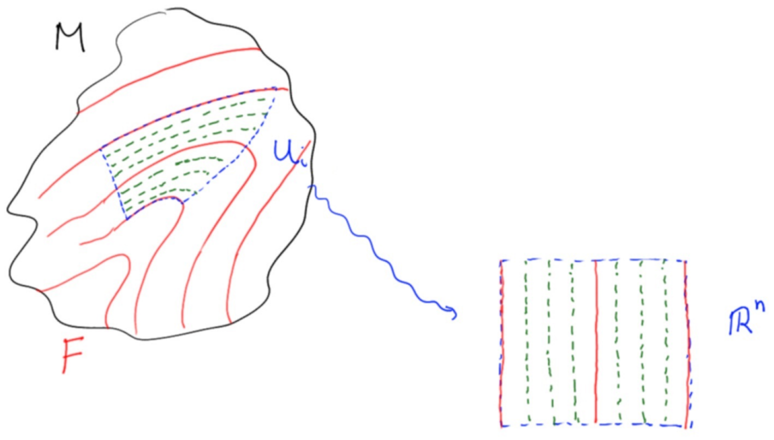
I.e. a solution of $\mathcal{G}$ on a complete transversal $T \subset M$, invariant under holonomy.

A **concordance** $(M \times [0,1], \mathcal{F}, \sigma)$ between $(M, \mathcal{F}_0, \sigma_0)$ and $(M, \mathcal{F}_1, \sigma_1)$ is

- a foliation $\mathcal{F}$ cutting the boundary transversely in $\mathcal{F}_i$.
- with transverse structure σ restricting to σ_i at the boundary.

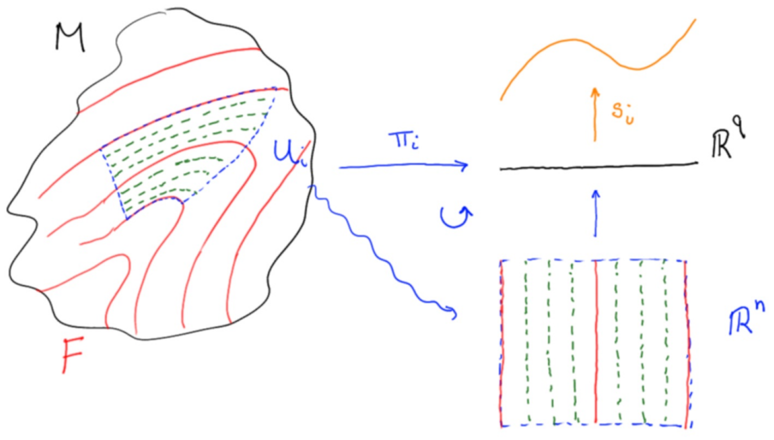
The groupoid picture

We cover $(M, \mathcal{F}, \sigma)$ with foliated charts $\{U_i\}$.



The groupoid picture

Let $\pi_i : U_i \to \mathbb{R}^q$. On each $\pi_i(U_i)$, we have a solution s_i of $\mathcal{G}(\pi_i(U_i))$.



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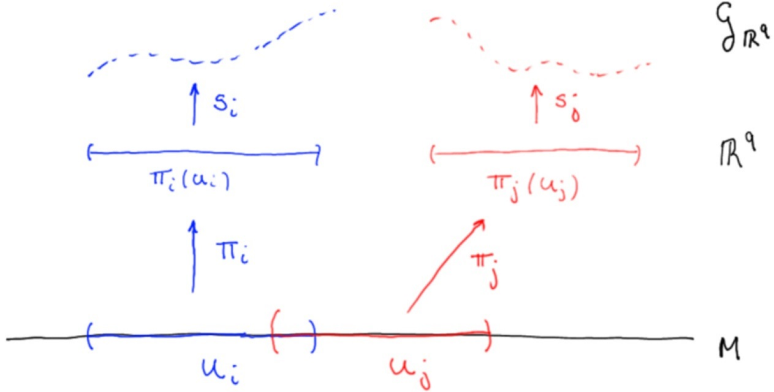


In $U_i \cap U_j$, the transition $\phi_{i,j}$ takes s_i to s_j.

The groupoid picture

From $(M, \mathcal{F}, \sigma)$ we have produced a 1-cocycle valued in the groupoid

$$\Gamma_{\mathcal{G}} := \Gamma_q \ltimes \mathcal{G}_{\mathbb{R}^q} \quad \rightrightarrows \quad \mathcal{G}_{\mathbb{R}^q}, \qquad \text{where:}$$

- $\mathcal{G}_{\mathbb{R}^q}$ is the étale space of $\mathcal{G}$ on $\mathbb{R}^q$ with canonical solution $\sigma_{\mathcal{G}}$.
- Γ_q is the étale groupoid of diffeomorphism germs of $\mathbb{R}^q$.

More intrinsically, our data is equivalent to the principal $\Gamma_{\mathcal{G}}$-bundle

$$\begin{array}{ccc} P & \circlearrowleft & \Gamma_{\mathcal{G}} \\ \downarrow & \overset{\mu}{\searrow} & \downdownarrows \\ M & & \mathcal{G}_{\mathbb{R}^q} \end{array}$$

of submersion germs $M \to \mathcal{G}_{\mathbb{R}^q}$ pulling back $\sigma_{\mathcal{G}}$ to $(\mathcal{F}, \sigma)$.

The groupoid picture

The formal analogue of $\Gamma_{\mathcal{G}}$ is

$$\Gamma^f_{\mathcal{G}} := (\Gamma_q)_0 \ltimes (\mathcal{G}_{\mathbb{R}^q})_0.$$

This is the stalk at the origin of the product of the sheaf of embeddings into $\mathbb{R}^q$ and the sheaf of solutions of $\mathcal{G}$ (both over $\mathbb{R}^q$). As such, it is endowed with the colimit topology coming from the Whitney topology on the sheaves.

There is a scanning morphism

$$\Gamma_{\mathcal{G}} \quad \longrightarrow \quad \Gamma^f_{\mathcal{G}}.$$

Some comments

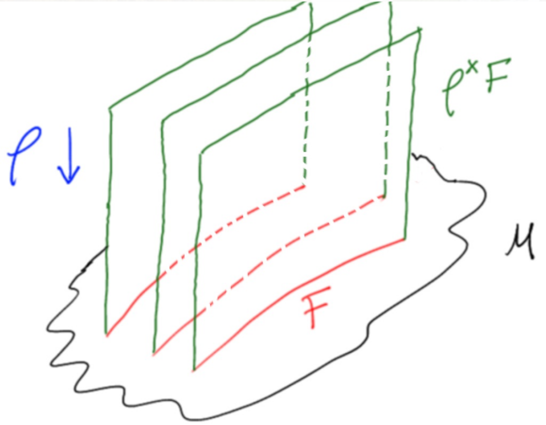
In applications, $\mathcal{G}$ is usually defined using finite jets. Then, $\Gamma^f_{\mathcal{G}}$ can be replaced by a finite-dimensional manifold of jets. This appeared already in the works of Haefliger and is central to the Geometry of PDEs approach of Crainic, Salazar, Yudilevich, Cattafi, and Accornero.

When the solutions of $\mathcal{G}$ have a local model, we can replace the base $\mathcal{G}_{\mathbb{R}^q}$ of the groupoid by $\mathbb{R}^q$ (endowed with the local model). The resulting groupoid is Morita equivalent to $\Gamma_{\mathcal{G}}$.

Haefliger structures

Submersions

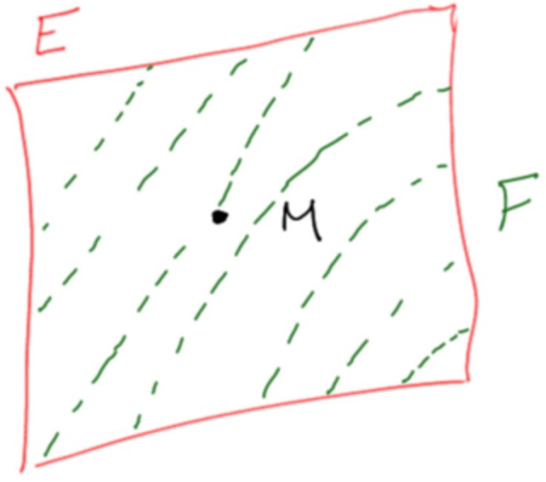
Consider a vector bundle $\rho : E \to (M, \mathcal{F}, \sigma)$.



The spaces $(M/\mathcal{F}, \sigma)$ and $(E/\rho^*\mathcal{F}, \rho^*\sigma)$ are isomorphic.

Foliations of large codimension

Conversely, given $\rho : (E, \mathcal{F}, \sigma) \to M$, with E vector bundle, we want to think of it as a foliation with transverse structure on M.



Two papers

The following works due to Haefliger:

- *Feuilletages sur les variétés ouvertes*. Topology 9 (1970), 183–194.
- *Homotopy and integrability*. Manifolds Amsterdam, Springer L.N.M. 197 (1970), 133–175.

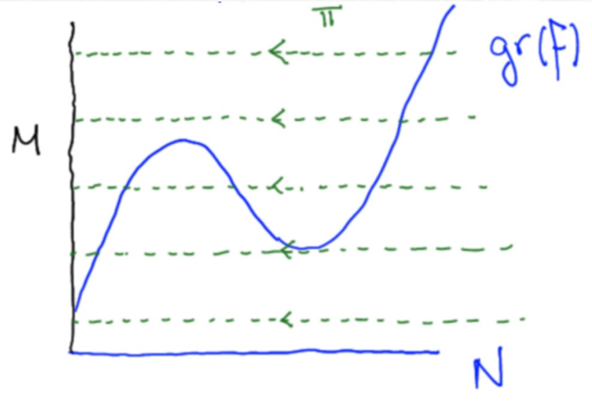
The issue: Given $f : N \to (M, \mathcal{F})$ arbitrary, there may be no pullback foliation $f^*\mathcal{F}$.

Haefliger's idea: Enlarge the class of foliations to another geometry ("**Haefliger structures**") that can always be pulled back.

Pullbacks

Given $f : N \to (M, \mathcal{F}, \sigma)$, consider the factorisation:

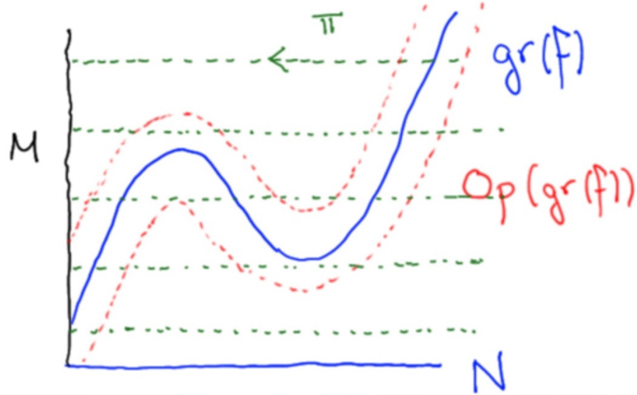
$$N \xrightarrow{\text{gr}(f)} N \times M \xrightarrow{\pi} (M, \mathcal{F}, \sigma).$$



Pullbacks

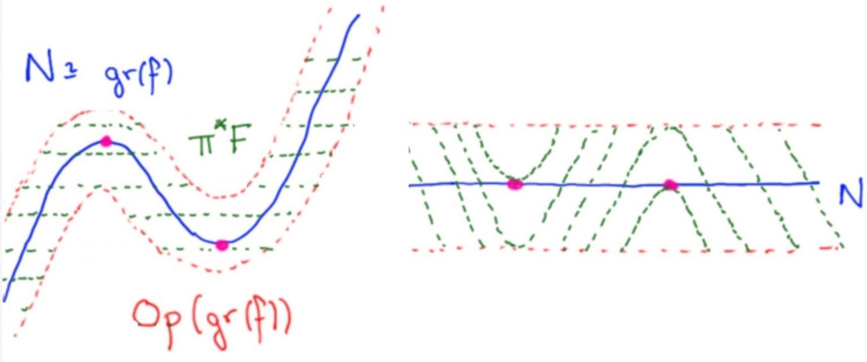
Given $f : N \to (M, \mathcal{F}, \sigma)$, consider the factorisation:

$$N \xrightarrow{\mathrm{gr}(f)} \mathrm{Op}(\mathrm{gr}(f)) \xrightarrow{\pi} (M, \mathcal{F}, \sigma).$$



Pullbacks

We define $f^*\mathcal{F} := (\mathrm{Op}(\mathrm{gr}(f)), \pi^*\mathcal{F}, \pi^*\sigma)$.



This is a Haefliger structure ("a singular foliation together with its desingularisation").

Haefliger structures

A **Haefliger structure with transverse $\mathcal{G}$-geometry** on M is a triple $(\pi : E \to M, \mathcal{F}, \sigma)$ where:

- E is a (germ of) bundle over M.
- $\mathcal{F} \subset TE$ is a foliation *transverse to the fibres of* π,
- σ is a solution of $\mathcal{G}$ transverse to $\mathcal{F}$.

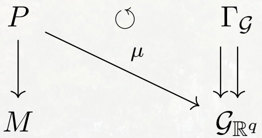
So one can pullback:

$$\begin{array}{ccc} (f^*E, \widetilde{f}^*\mathcal{F}, \widetilde{f}^*\sigma) & \xrightarrow{\widetilde{f}} & (E, \mathcal{F}, \sigma) \\ \downarrow{\scriptstyle f^*\pi} & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{f} & M. \end{array}$$

The groupoid picture

Given $(\pi : E \to M, \mathcal{F}, \sigma)$ we can produce a 1-cocycle with values in $\Gamma_{\mathcal{G}}$. Which can be restricted to M.

This yields again the principal $\Gamma_{\mathcal{G}}$-bundle:



of germs of mapping $M \to \mathcal{G}_{\mathbb{R}^q}$ pulling back the tautological solution to $(E, \mathcal{F}, \sigma)$.

Now μ is submersive if and only if $\mathcal{F}|_M$ is a foliation.

Some classic results

Haefliger's theorem

Two Haefliger structures on M are **concordant** if there is a Haefliger structure on $M \times [0,1]$ restricting to them at the boundary.

We write $\mathrm{Haef}_{\mathcal{G}}(M)$ for the set of Haefliger structures on M with transverse $\mathcal{G}$-geometry.

There is a **classifying space**

$$B\Gamma_{\mathcal{G}} \quad := \quad \mathcal{G}(\mathbb{R}^q) \cup_{s,t} \Delta^1 \times \Gamma_{\mathcal{G}} \cup \Delta^2 \times \Gamma_{\mathcal{G}}^{(2)} \ldots$$

such that $\mathrm{Haef}_{\mathcal{G}}(M)/_{\text{concordance}} \cong [M, B\Gamma_{\mathcal{G}}]$.

This relates geometry (classifying foliations) to topology (what is the homotopy type of $B\Gamma_{\mathcal{G}}$?).

Some key results

Theorem (Haefliger, Thurston)

$\mathrm{Haef}_q(M)/_{\text{concordance}} \cong \mathrm{Fol}_q(M)/_{\text{concordance}}$.

To be precise, on the left hand side we also need to keep track of a monomorphism mapping the normal bundle of the Haefliger structure to TM.

Theorem (Thurston)

Any hyperplane field is homotopic to a foliation.

Any map into $B\mathrm{GL}_1$ can be lifted to $B\Gamma_1$. Here $B\mathrm{GL}_1$ classifies the normal bundle.

Some key results

Theorem (Haefliger, Mather, Thurston)

Any 2-plane field is homotopic to a foliation.

$B\Gamma_q \to B\mathrm{GL}_q$ is $(q+2)$-connected.

Theorem (Thurston)

$B\Gamma_q \to B\mathrm{GL}_q$ is at most $(2q+2)$-connected.

This follows from the Godbillon-Vey invariant.

Conjecture (Haefliger-Thurston)

$B\Gamma_q \to B\mathrm{GL}_q$ is $(2q+1)$-connected.

This is supported by the vanishing of the differentiable cohomology in this range.

What about transverse geometries?

Theorem (McDuff)

Line fields with transverse volume are flexible.

$B\Gamma_{\mathrm{vol},q} \to B\mathrm{SL}_q^{\text{mon}} \times K(\mathbb{R}, q)$ is $(q+1)$-connected.

Theorem (McDuff)

Line fields with transverse symplectic structure are flexible.

$B\Gamma_{\mathrm{symp},2n} \to B\mathrm{Sp}_n^{\text{mon}} \times K(\mathbb{R}, 2)$ is $(2n+1)$-connected.

Theorem (McDuff, Nariman)

Rank-2, transversely contact, Haefliger structures are flexible.

$B\Gamma_{\mathrm{cont},2n+1} \to B\mathrm{Sp}_n^{\text{mon}}$ is $(2n+3)$-connected.

What about transverse geometries?

Open question

How are $\mathrm{Fol}_{\mathcal{G}}(M)/_{\text{concordance}}$ and $\mathrm{Haef}_{\mathcal{G}}(M)/_{\text{concordance}}$ related?

- On the Haefliger side, we need to include a monomorphism of the normal bundle into TM as part of the data.
- Haefliger's theorem says that there is a 1-to-1 correspondence in open manifolds.
- In closed manifolds, whether we can regularise depends on the concrete geometry. For instance: due to the lack of h-principle for symplectic structures, we know that it is in general impossible to regularise a transversely symplectic foliation.

Theorem I

Statement

Let $\mathcal{G}$ be open, Diff-invariant, of dim q.

Theorem (Gromov)

$B\Gamma_{\mathcal{G}} \to B\Gamma^f_{\mathcal{G}}$ is $(q-1)$-connected.

In particular: Every formal solution of $\mathcal{G}$ over M can be homotoped to a solution over a submanifold of positive codimension.

Theorem I (Fokma, dP, Toussaint)

$B\Gamma_{\mathcal{G}} \to B\Gamma^f_{\mathcal{G}}$ is q-connected.

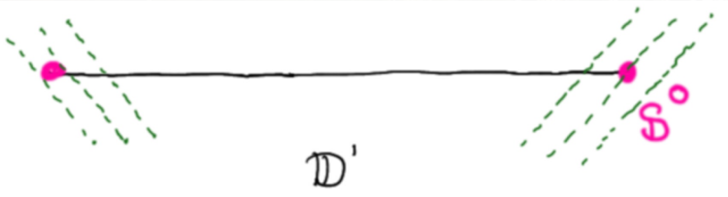
In particular: Every formal solution of $\mathcal{G}$ over M can be homotoped to a global solution with singularities.

Idea

Theorem I (Fokma, dP, Toussaint)

$B\Gamma_{\mathcal{G}} \to B\Gamma_{\mathcal{G}}^{f}$ is q-connected.

Consider $\mathcal{G}$ to be "functions with positive slope in $\mathbb{R}$". I.e. $q = 1$.

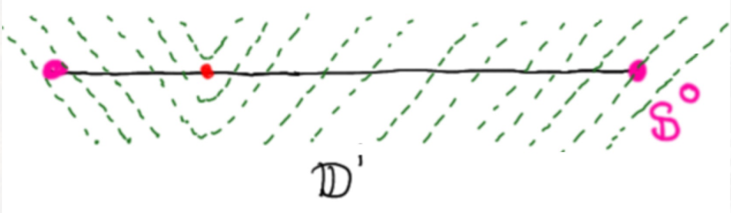


Idea

Theorem I (Fokma, dP, Toussaint)

$B\Gamma_{\mathcal{G}} \to B\Gamma^f_{\mathcal{G}}$ is q-connected.

First step: Use smooth case to extend the foliation to the interior.

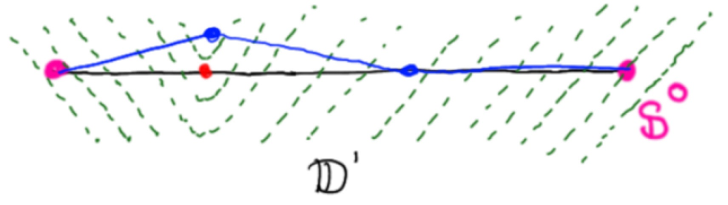


Idea

Theorem I (Fokma, dP, Toussaint)

$B\Gamma_{\mathcal{G}} \to B\Gamma^{f}_{\mathcal{G}}$ is q-connected.

Second step: Use Thurston's jiggling to replace $\mathbb{D}^1$ by a piecewise transverse version.

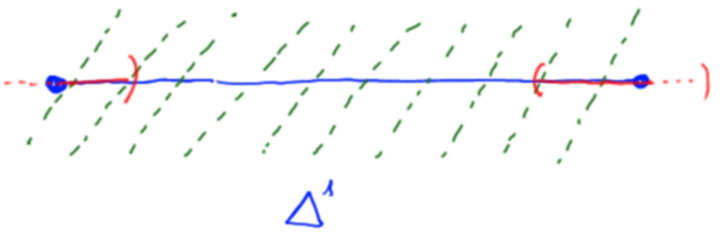


Idea

Theorem I (Fokma, dP, Toussaint)

$B\Gamma_{\mathcal{G}} \to B\Gamma^{f}_{\mathcal{G}}$ is q-connected.

Third step: Over each 0-simplex, put increasing function.

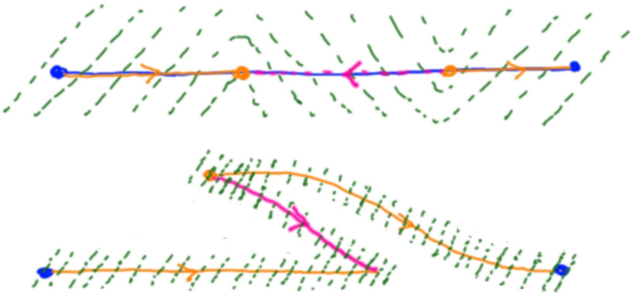


Idea

Theorem I (Fokma, dP, Toussaint)

$B\Gamma_{\mathcal{G}} \to B\Gamma^{f}_{\mathcal{G}}$ is q-connected.

Fourth step: Over each 1-simplex, use wrinkling.



More generally...

Theorem I (Fokma, dP, Toussaint)

$B\Gamma_{\mathcal{G}} \to B\Gamma^{f}_{\mathcal{G}}$ is q-connected.

- We have a (singular) solution of $\mathcal{G}$ along $\mathbb{S}^{q-1}$, which we want to extend to $\mathbb{D}^q$.
- Use Γ_q case.
- Use jiggling to work with simplices transverse to foliation.
- Apply holonomic approximation along the codimension-1 skeleton.
- Use holonomic approximation parametrically to wrinkle in top-cells.

Closing comments

Main punchline: Haefliger structures provide a nice setup to phrase wrinkling for arbitrary Diff-invariant, open relations $\mathcal{G}$.

- Theorem I applies to any microflexible sheaf on the site Man.
- Generalises wrinkled submersions (Eliashberg, Mishachev), universal hole for contact structures (Borman, Eliashberg, Murphy), folded symplectic structures (Cannas da Silva).
- Addresses a programme put forward by Laudenbach-Meigniez.

Previous work of dP-Toussaint dealt with horizontal manifolds in jet space, seen as “generalised solutions”; this was a generalisation of wrinkled *embeddings*. Theorem I is the “étale space” counterpart (but the proofs are quite different).

Theorem II

Idea

Let $\mathcal{G}$ be "immersions of q-dimensional manifolds into a fixed target N of dimension greater than q".

Theorem II (Fokma, dP, Toussaint)

$B\Gamma_{\mathcal{G}} \to B\Gamma^f_{\mathcal{G}}$ is not $q+1$-connected.

In fact: two solutions of $\mathcal{G}$ with disjoint image cannot be concordant.

Helpful observation

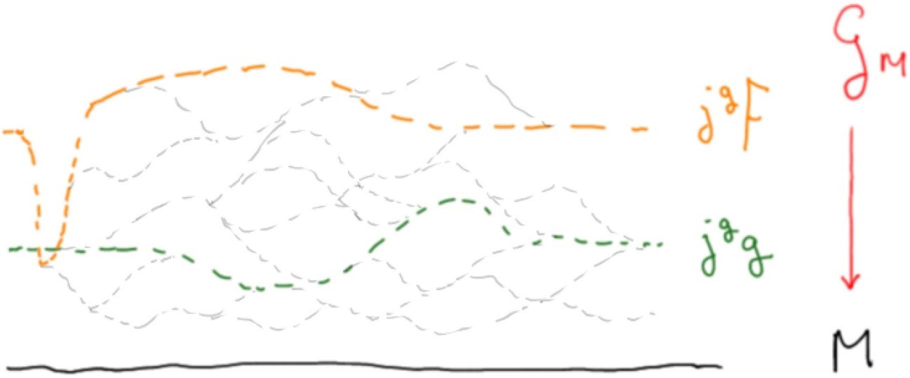
Two sections of $\mathcal{G}_M$ are homotopic (as maps) iff they are the same.

The observation is true for étale spaces of arbitrary sheaves.

Proof of observation

Observation

Two sections of $\mathcal{G}_M$ are homotopic (as maps) iff they are the same.



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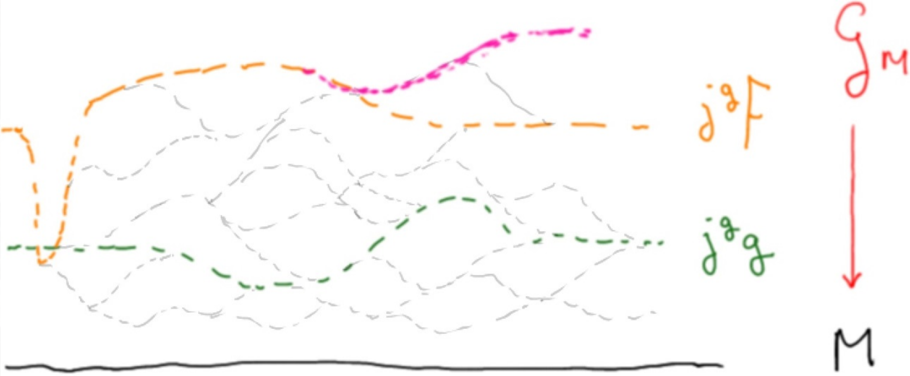

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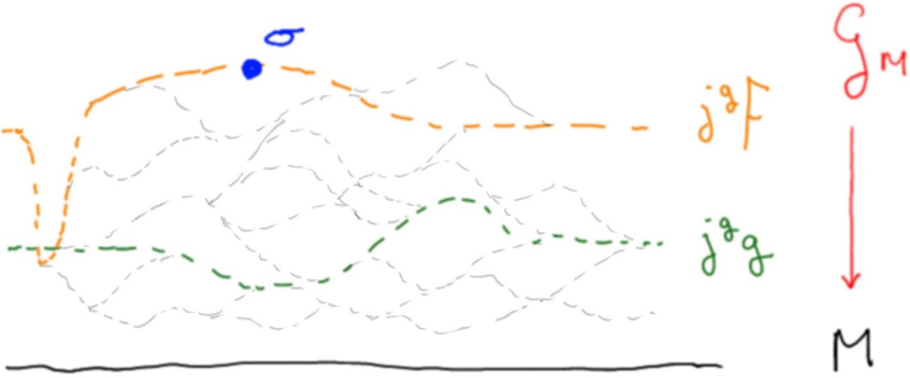

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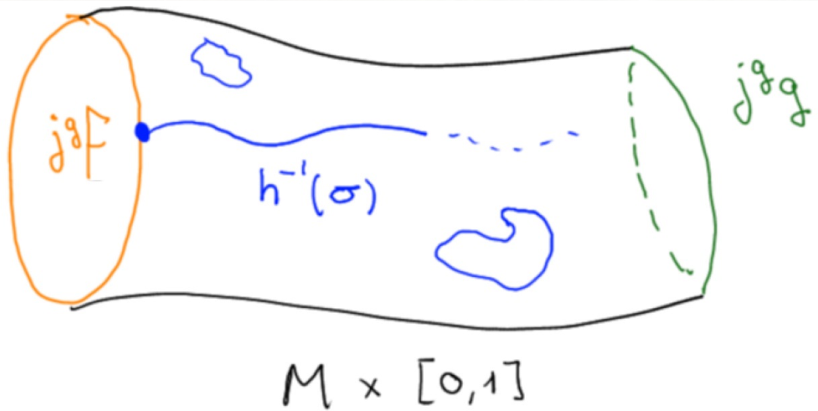
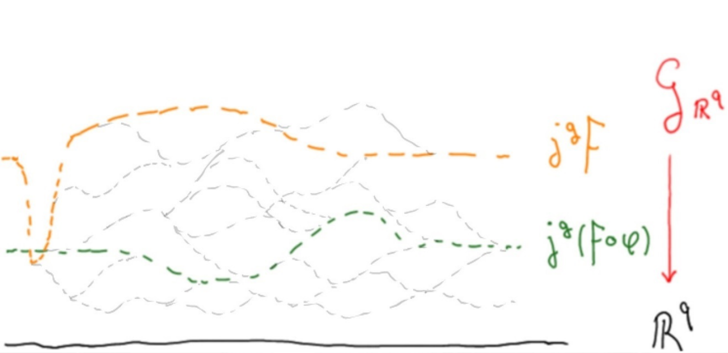


Proof of the theorem

Theorem II (Fokma, dP, Toussaint)

Two solutions of $\mathcal{G}$ with disjoint image are not concordant.

Γ_q acting on $\mathcal{G}_{\mathbb{R}^q}$ identifies diffeomorphic solutions.

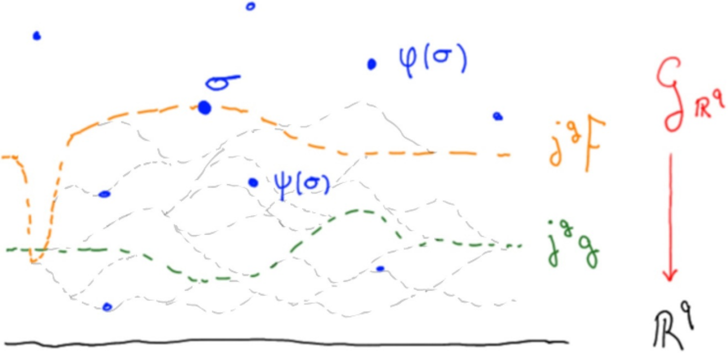


Proof of the theorem

Theorem II (Fokma, dP, Toussaint)

Two solutions of $\mathcal{G}$ with disjoint image are not concordant.

Let us consider a germ σ in one of them but not the other.

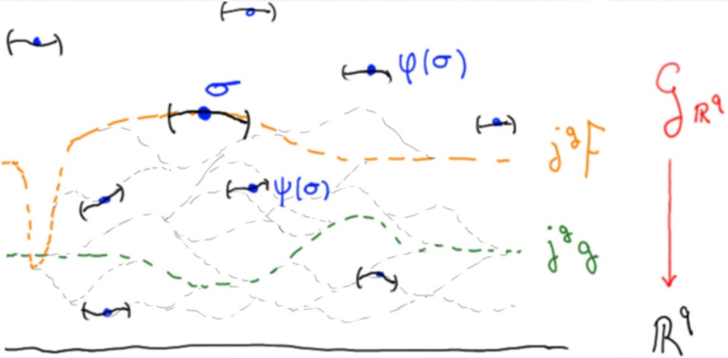


Proof of the theorem

Theorem II (Fokma, dP, Toussaint)

Two solutions of $\mathcal{G}$ with disjoint image are not concordant.

The germ σ is “isolated” (no diffeomorphic germs nearby).

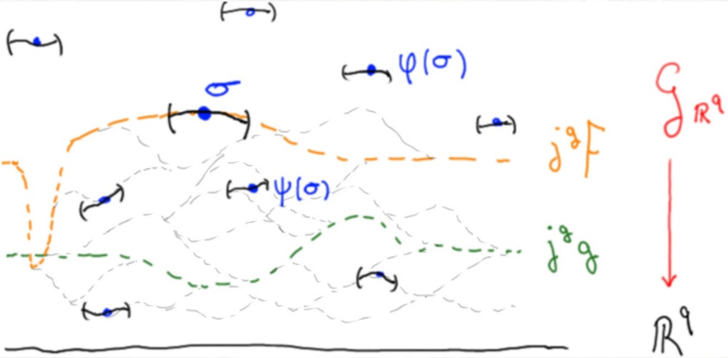


Proof of the theorem

Theorem II (Fokma, dP, Toussaint)

Two solutions of $\mathcal{G}$ with disjoint image are not concordant.

There is then a pairing between $\Gamma_q(\sigma)$ and $M \to B\Gamma_{\mathcal{G}}$.

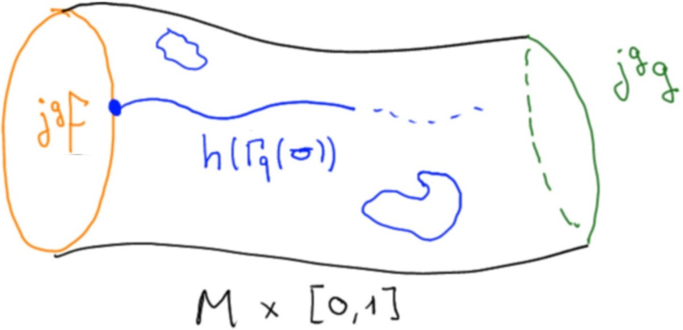


Proof of the theorem

Theorem II (Fokma, dP, Toussaint)

Two solutions of $\mathcal{G}$ with disjoint image are not concordant.

Solutions pairing differently with $\Gamma_q(\sigma)$ cannot be homotopic (as maps into classifying space).



Some concrete examples

Theorem II (Fokma, dP, Toussaint)

Two solutions of $\mathcal{G}$ with disjoint image are not concordant.

Concrete examples:

- Immersions.
- **To work out:** Functions.
- **To work out:** “Generic” geometries (e.g. generic metrics).

Closing questions

To be done: Work out examples in detail, address higher homotopy groups.

Open question: What about geometries that are a bit flexible (e.g. they have a local model)?

Open question: The projection from germs to jets relates homotopy classes (in the étale topology) to horizontal homotopy classes. What is the connectivity of these maps?

Open question: Is there an interesting concordance theory for other groupoids? For (some subclass of) singular foliations? For Poisson structures?

Thank You!