Multiplicative multivector fields and forms on Lie groupoids

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- Geometric structures on a Lie groupoid that are compatible with the groupoid multiplication are called multiplicative structures.
- Related notions: Poisson groupoids, multisymplectic groupoids, Dirac groupoids, Glanon groupoids, Pfaffian groupoids, quasi-Hamiltonian groupoids, etc.

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- Related notions: Poisson groupoids, multisymplectic groupoids, Dirac groupoids, Glanon groupoids, Pfaffian groupoids, quasi-Hamiltonian groupoids, etc.
- Universal lifting theorems or Lie theory:
  multiplicative structures on Lie groupoids
  infinitesimal multiplicative structures on Lie algebroids.

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Plans:

1. Multiplicative multivector fields and characteristic pairs
2. Multiplicative forms and characteristic pairs
3. Multiplicative forms on Poisson groupoids
4. Multiplicative forms on quasi-Poisson groupoids

On the reduced space of multiplicative multivectors, arXiv: 2003. 13384.

Multiplicative forms on Poisson groupoids, arXiv: 2201.06242.

Multiplicative forms on quasi-Poisson groupoids, in progress.

Multiplicative k-vector fieds

A k-vector field $\Pi \in \mathfrak{X}^k(G)$ on a Lie group G is multiplicative, if

$$\Pi_{gr} = L_{g*}\Pi_r + R_{r*}\Pi_g, \qquad \forall g, r \in G, \qquad (\text{or} \qquad m_*(\Pi \times \Pi) = \Pi).$$

$\Pi|_e = 0$.

Example

For $\tau \in \wedge^k \mathfrak{g}$, $\overrightarrow{\tau} - \overleftarrow{\tau} \in \mathfrak{X}^k(G)$ is multiplicative.

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For $\tau \in \wedge^k \mathfrak{g}$, $\overrightarrow{\tau} - \overleftarrow{\tau} \in \mathfrak{X}^k(G)$ is multiplicative.

A Poisson Lie group is a Lie group with a multiplicative bivector field Π such that $[\Pi, \Pi] = 0$.

Three equivalent definitions

Definition

A k-vector field $\Pi \in \mathfrak{X}^k(\mathcal{G})$ is **multiplicative***, if the graph of groupoid multiplication is a coisotropic submanifold in $\mathcal{G} \times \mathcal{G} \times \mathcal{G}$ with respect to $\Pi \times \Pi \times (-1)^{k-1}\Pi$.*

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Definition 2

1. $\Pi_{gr} = L_{b_g *}\Pi_r + R_{b'_r *}\Pi_g - L_{b_g *}R_{b'_r *}\Pi_x$, where b_g and b'_r are bisections passing g and r. (iff $[\Pi, \overrightarrow{u}]$ is right-invariant, denoted by $\overrightarrow{\delta_\Pi(u)}$, thus $\delta_\Pi : \Gamma(A) \to \Gamma(\wedge^k A)$.)
2. for any $\xi \in \Omega^1(M)$, $\iota_{t^*(\xi)}\Pi$ is right-invariant ($\overrightarrow{\delta_\Pi(f)} = [\Pi, t^* f]$.)
3. M is a coisotropic submanifold of $\mathcal{G}$. ($\Pi|_M(\xi_1, \cdots, \xi_k) = 0, \forall \xi_i \in A^*$.)

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Definition 3

$\Pi^\sharp$ is a groupoid morphism:

$$\begin{array}{ccc} \oplus^{k-1} T^*\mathcal{G} & \xrightarrow{\Pi^\sharp} & T\mathcal{G} \\ \downdownarrows & & \downdownarrows \\ \oplus^{k-1} A^* & \xrightarrow{\pi^\sharp} & TM \end{array}.$$

Lemma

For a Lie group G and any $\Pi \in \mathfrak{X}^k_{\mathrm{mult}}(G)$, there exists $c \in Z^1(G, \wedge^k \mathfrak{g})$ such that

$$\Pi_g = R_{g*} c(g), \qquad g \in G.$$

$\Pi|_e = 0$.

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$\Pi|_e = 0$.

Proposition (Chen-Stienon-Xu)

If $\Pi \in \mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G})$, denote by $\pi = \mathrm{pr}_{\Gamma(TM \otimes (\wedge^{k-1} A))} \Pi|_M$, then

$$\begin{aligned}\Pi|_M &= B\pi := \frac{1 - e^{-D_\rho}}{D_\rho}(\pi) \\ &= \pi - \tfrac{1}{2!} D_\rho \pi + \tfrac{1}{3!} D_\rho^2 \pi + \cdots + \tfrac{(-1)^{k-1}}{k!} {D_\rho}^{k-1} \pi \in \wedge^k (TM \oplus A) / \wedge^k A,\end{aligned}$$

where $D_\rho(X + u) = \rho(u)$.

Characteristic pairs

$$1 \to \mathcal{H} \to \mathfrak{J}\mathcal{G} \to \mathcal{G} \to 1, \qquad 0 \to T^*M \otimes A \to \mathfrak{J}A \to A \to 0.$$

Definition

A characteristic pair of $(k,0)$-type is a pair

$$(c, \pi) \in Z^1(\mathfrak{J}\mathcal{G}, \wedge^k A) \times \Gamma(TM \otimes (\wedge^{k-1} A)),$$

such that, for all $\xi, \eta \in \Omega^1(M), [h_x] \in \mathcal{H}_x, x \in M$,

$$\begin{aligned} \iota_{\rho^*\xi}\iota_\eta \pi &= -\iota_{\rho^*\eta}\iota_\xi \pi, && (\ \rho\text{-compatible}) \\ c([h_x]) &= (B\pi)_x - L_{[h_x]}(B\pi)_x && (:= M_\pi([h_x]), M_\pi \in Z^1(\mathcal{H}, \wedge^k A)). \end{aligned}$$

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Theorem (Chen-L-Liu)

There is a 1-1 correspondence between multiplicative k-vector fields Π, and characteristic pairs (c,π) of $(k,0)$-type on $\mathcal{G}$ such that

$$\Pi_g = R_{g*}c([b_g]) + L_{b_g*}(B\pi)_x$$

holds for all $g \in \mathcal{G}$ and bisection b_g passing g and $x = s(g)$.

Example

For $\tau \in \Gamma(\wedge^k A)$, $\overrightarrow{\tau} - \overleftarrow{\tau} \in \mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G})$ *corresponds to* $(c = -d_{\mathfrak{J}\mathcal{G}}\tau, \pi = \rho \circ \tau)$, *where*

$$d_{\mathfrak{J}\mathcal{G}} : \wedge^k A \to C^1(\mathfrak{J}\mathcal{G}, \wedge^k A).$$

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Define two quotient space

$$\mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G})/\sim, \qquad R^k_\rho := \{[\pi]\},$$

where $\Pi \sim \Pi + \overleftarrow{\tau} - \overrightarrow{\tau}$ and $\pi \sim \pi + \rho \circ \tau$, where $\pi \in \Gamma(TM \otimes \wedge^{k-1} A)$ is ρ-compatible for any $\tau \in \wedge^k A$.

Classification theorem of k-vector fields

Theorem (Chen-L-Liu)

The quotient space of multiplicative k-vectors $\mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G})/\sim$ is the pullback of maps I^* and U. I.e., the diagram

$$\begin{array}{ccc} \mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G})/\sim & \xrightarrow{I^!} & R^k_\rho \\ \Big\downarrow K & & \Big\downarrow U \\ \mathrm{H}^1(\mathfrak{J}\mathcal{G}, \wedge^k A) & \xrightarrow{I^*} & \mathrm{H}^1(\mathcal{H}, \wedge^k A) \end{array}$$

is commutative and the map

$$\begin{array}{rccc} \Theta: & \mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G})/\sim & \xrightarrow{\cong} & \mathrm{H}^1(\mathfrak{J}\mathcal{G}, \wedge^k A)\ {}_{I^*}\!\times_U R^k_\rho \\ & [\Pi = (c, \pi)] & \mapsto & ([c], [\pi]) \end{array}$$

is an isomorphism.

Transitive case

For a transitive groupoid $\mathcal{G}$ over M, define

$$\mathcal{Q}^k := Z^1(\mathcal{G}, \wedge^k \ker \rho) \times \Gamma(\wedge^k A).$$

Two pairs $(\mathcal{F}, \Lambda) \sim (\mathcal{F}', \Lambda')$, if there exists some $\nu \in \Gamma(\wedge^k \ker\rho)$ s.t.

$$\mathcal{F}' = \mathcal{F} + d_{\mathcal{G}}\nu, \qquad \Lambda' = \Lambda + \nu.$$

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Theorem

We have a 1-1 correspondence

$$\begin{array}{ccc} \mathcal{Q}^k/_\sim & \cong & \mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G}), \\ [(\mathcal{F}, \Lambda)] & \mapsto & \overrightarrow{\mathcal{F}(\cdot)} + \overrightarrow{\Lambda} - \overleftarrow{\Lambda}. \end{array}$$

Moreover,

$$\mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G})/_\sim \cong H^1(\mathcal{G}, \wedge^k \ker\rho).$$

A graded Lie 2-algebra structure

For $\tau \in \Gamma(\wedge^l A)$, $\overrightarrow{\tau} - \overleftarrow{\tau} \in \mathfrak{X}^l_{\mathrm{mult}}(\mathcal{G})$.

Proposition (Bonechi-Ciccoli-Laurent-Gengoux-Xu)

There is a natural strict graded Lie 2-algebra on the complex

$$\Gamma(\wedge^\bullet A) \to \mathfrak{X}^\bullet_{\mathrm{mult}}(\mathcal{G}), \qquad \tau \mapsto \overrightarrow{\tau} - \overleftarrow{\tau}.$$

$$[\Pi, \overrightarrow{\tau} - \overleftarrow{\tau}] = \overrightarrow{\delta_\Pi \tau} - \overleftarrow{\delta_\Pi \tau}.$$

$$[\Pi, \tau] = \delta_\Pi(\tau) \in \Gamma(\wedge^{k+l-1} A), \qquad \Pi \in \mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G}), \tau \in \Gamma(\wedge^l A).$$

See also Berwick-Evans-Lerman, Ortiz-Waldron's work.

Lie 2-algebras

A Lie 2-algebra (Baez-Crans) is a 2-term L_∞-algebra (Schlessinger-Stasheff). It has the data

- $d : \mathfrak{g}_{-1} \to \mathfrak{g}_0$;
- the 2-bracket $[\cdot,\cdot]_2 : \mathfrak{g}_0 \wedge \mathfrak{g}_i \to \mathfrak{g}_i, i = 0, -1$;
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s.t., for all $w, x, y, z \in \mathfrak{g}_0$ and $u, v \in \mathfrak{g}_{-1}$,

(1) $[du, v]_2 = -[dv, u]_2, \qquad d[x, u]_2 = [x, du]_2$;

(2) $[[x, y]_2, z]_2 + [[y, z]_2, x]_2 + [[z, x]_2, y]_2 = d[x, y, z]_3$;

(3) $[[x, y]_2, u]_2 + [[y, u]_2, x]_2 + [[u, x]_2, y]_2 = [x, y, du]_3$;

(4) "$d_{CE}[\cdot,\cdot,\cdot]_3 = 0$", i.e.,

$$\begin{aligned} & -[w, [x, y, z]_3]_2 - [y, [x, z, w]_3]_2 + [z, [x, y, w]_3]_2 + [x, [y, z, w]_3]_2 \\ = \; & [[x, y]_2, z, w]_3 - [[x, z]_2, y, w]_3 + [[x, w]_2, y, z]_3 + [[y, z]_2, x, w]_3 \\ & -[[y, w]_2, x, z]_3 + [[z, w]_2, x, y]_3. \end{aligned}$$

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If $[\cdot,\cdot,\cdot]_3 = 0$, then it is called a strict Lie 2-algebra.

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\begin{aligned}
& -[w, [x, y, z]_3]_2 - [y, [x, z, w]_3]_2 + [z, [x, y, w]_3]_2 + [x, [y, z, w]_3]_2 \\
= \quad & [[x, y]_2, z, w]_3 - [[x, z]_2, y, w]_3 + [[x, w]_2, y, z]_3 + [[y, z]_2, x, w]_3 \\
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\end{aligned}
$$

If $[\cdot,\cdot,\cdot]_3 = 0$, then it is called a strict Lie 2-algebra.

We always denote by $\triangleright := [\cdot,\cdot]_2 : \mathfrak{g}_0 \wedge \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$.

Definition

Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Lie 2-algebras. A **Lie 2-algebra homomorphism** consists of

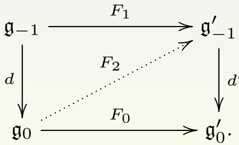
- a chain map $F_0 : \mathfrak{g}_0 \to \mathfrak{g}'_0$, $F_1 : \mathfrak{g}_{-1} \to \mathfrak{g}'_{-1}$ such that $F_0 \circ d = d' \circ F_1$,
- a skew-symmetric bilinear map $F_2 : \wedge^2 \mathfrak{g}_0 \to \mathfrak{g}'_{-1}$,

such that, for $x, y, z \in \mathfrak{g}_0$ and $u \in \mathfrak{g}_{-1}$.

(1) $F_0[x,y]_2 - [F_0(x), F_0(y)]'_2 = d' F_2(x,y)$,

(2) $F_1[x,u]_2 - [F_0(x), F_1(u)]'_2 = F_2(x, d(u))$,

(3) $F_1[x,y,z]_3 - [F_0(x), F_0(y), F_0(z)]'_3 = [F_0(x), F_2(y,z)]'_2 - F_2([x,y]_2, z) + c.p.$.



Multiplicative forms and characteristic pairs

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A k-form $\Theta \in \Omega^k(\mathcal{G})$ on a Lie groupoid $\mathcal{G}$ is multiplicative if

$$m^*\Theta = \mathrm{pr}_1^*\Theta + \mathrm{pr}_2^*\Theta,$$

where $m, \mathrm{pr}_i : \mathcal{G}^{(2)} \to \mathcal{G}, i = 1, 2$ are the multiplication and the projections.

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Equivalently,

(1) The graph of multiplication $\{(g, r, gr); s(g) = t(r)\}$ is an isotropic submanifold of $\mathcal{G} \times \mathcal{G} \times \mathcal{G}$ w.r.t $\Theta \oplus \Theta \oplus \Theta$;

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Example

For a Lie group G, we have $\Omega^k_{\mathrm{mult}}(G) = 0, k \geq 2$.

$$\Theta(X_{gr}, Y_{gr}) = \Theta(R_{r*}\tilde{X}_g, L_{g*}\tilde{Y}_r) = \Theta(\tilde{X}_g \cdot 0_r, 0_g \cdot \tilde{Y}_r) = \Theta(\tilde{X}_g, 0) + \Theta(0, \tilde{Y}_r) = 0.$$

Example

If V is a vector space,

$$\mathfrak{X}^k_{\mathrm{mult}}(V) = \mathrm{Hom}(V, \wedge^k V), \qquad \Omega^1_{\mathrm{mult}}(V) = V^*.$$

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Example

A vector bundle $E \to M$ is a Lie groupoid. Multiplicative multivector fields and forms are indeed linear multivector fields and forms. So $\Pi \in \mathfrak{X}^k_{\mathrm{mult}}(E)$ if it is locally of the form

$$\Pi = \frac{1}{k!}\Pi^{i_1\cdots i_k}_j(q)p^j \frac{\partial}{\partial p^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial p^{i_k}} + \frac{1}{(k-1)!}\Pi^{i_1\cdots i_{k-1},j}(q)\frac{\partial}{\partial p^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial p^{i_{k-1}}}\wedge\frac{\partial}{\partial q^j}.$$

Similarly, a k-form $\Theta \in \Omega^k_{\mathrm{mult}}(E)$ if

$$\Theta = \frac{1}{k!}\Theta_{i_1\cdots i_k,j}(q)p^j dq^{i_1}\wedge\cdots\wedge dq^{i_k} + \frac{1}{(k-1)!}\Theta_{i_1\cdots i_{k-1},j}(q)dq^{i_1}\wedge\cdots dq^{i_{k-1}}\wedge dp^j,$$

where $\{q^i\}$ are the coordinates on M, and $\{p^j\}$ are coordinates of the fiber.

Lemma

For $\Theta \in \Omega^k_{\mathrm{mult}}(\mathcal{G})$, *let* $\theta = \mathrm{pr}_{\Gamma(A^*\otimes(\wedge^{k-1}T^*M))}\Theta|_M$. *Then*

$$\Theta|_M = \frac{e^{D_{\rho^*}} - 1}{D_{\rho^*}}(\theta) = \theta + \frac{1}{2}D_{\rho^*}\theta + \frac{1}{3!}D^2_{\rho^*}\theta + \cdots + \frac{1}{k!}D^{k-1}_{\rho^*}\theta, \qquad (=: \mathrm{B}\theta)$$

where $D_{\rho^*}(\xi + \chi) = \rho^*\xi$.

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where $D_{\rho^*}(\xi + \chi) = \rho^*\xi$*.*

$$A \leftrightarrow T^*M, \qquad TM \leftrightarrow A^*.$$

Definition

A characteristic pair of $(0, k)$*-type is a pair* (e, θ) *with*

$$e \in Z^1(\mathfrak{J}\mathcal{G}, \wedge^k T^*M), \qquad \theta \in \Gamma(A^* \otimes (\wedge^{k-1} T^*M)),$$

satisfying

$$\begin{aligned} \iota_u \iota_{\rho(v)}\theta &= -\iota_v \iota_{\rho(u)}\theta, && \rho - \textit{compatible}, \\ e([h]) &= R_h^*(B\theta) - B\theta, && [h] \in \mathcal{H}, \\ \rho^* \circ e &= -d_{\mathfrak{J}\mathcal{G}}\theta. \end{aligned}$$

Theorem

There is a 1-1 correspondence between multiplicative k-forms $\Theta \in \Omega^k_{\mathrm{mult}}(\mathcal{G})$ and characteristic pairs (e, θ) of $(0, k)$–type, s.t.

$$\Theta_g = R^*_{b_g^{-1}}\big(e([b_g]) + B\theta_y\big),$$

where b_g is a local bisection through passing g and $y = t(g)$.

This result is a reformulation of Proposition 4.1 in

M. Crainic, M. A. Salazar and I. Struchiner, Multiplicative forms and Spencer operators, *Math. Z.* 279 (2015), no. 3-4, 939-979.

Theorem

There is a 1-1 correspondence between multiplicative k-forms $\Theta \in \Omega^k_{\text{mult}}(\mathcal{G})$ *and characteristic pairs* (e, θ) *of* $(0, k)$*–type, s.t.*

$$\Theta_g = R^*_{b_g^{-1}}\big(e([b_g]) + B\theta_y\big),$$

where b_g *is a local bisection through passing* g *and* $y = t(g)$*.*

This result is a reformulation of Proposition 4.1 in

M. Crainic, M. A. Salazar and I. Struchiner, Multiplicative forms and Spencer operators, *Math. Z.* 279 (2015), no. 3-4, 939-979.

Example

In the group case, the characteristic pair of $\Theta \in \Omega^1_{\text{mult}}(G)$ *is* $(0, \theta)$*, where* $\theta \in \mathfrak{g}^{*G}$ *and* $\Theta_g = R^*_{g^{-1}}\theta$*.*

Example

Let $\gamma \in \Omega^k(M)$*. Then* $s^*\gamma - t^*\gamma \in \Omega^k_{\text{mult}}(\mathcal{G})$ *and its characteristic pair is* $(d_{\mathfrak{J}\mathcal{G}}\gamma, -\rho^*\gamma)$*.*

Transitive case

Proposition

Let $\mathcal{G}$ be a transitive Lie groupoid over M.

(1) *If $k \geqslant 2$, then all multiplicative k-forms are of the form $s^*\gamma - t^*\gamma$ for $\gamma \in \Omega^k(M)$;*

(2) *All multiplicative 1-forms Θ on $\mathcal{G}$ are determined by some $\theta \in \Gamma(A^*)$ such that $\iota_a(d_{\mathfrak{J}\mathcal{G}}\theta) = 0$ for $a \in \ker\rho$ in the way*

$$\Theta_g(R_{[b_g]}(u + X)) = \theta_{t(g)}(u + v) - \theta_{s(g)}(\mathrm{Ad}_{[b_g]^{-1}} v), \qquad \rho(v) = X.$$

Multiplicative forms on Poisson groupoids

Definition

A Poisson groupoid is a groupoid $\mathcal{G}$ with a $P \in \mathfrak{X}^2_{\text{mult}}(\mathcal{G})$ such that $[P, P] = 0$.

On $\Omega^1(\mathcal{G})$, we have a Lie bracket:

$$[\alpha, \beta]_P = L_{P^\sharp\alpha}\beta - L_{P^\sharp\beta}\alpha - dP(\alpha, \beta).$$

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Extending to all forms, one obtain the **Koszul bracket** on $\Omega^\bullet(\mathcal{G})$:

$$[\alpha, \beta]_P = (-1)^{k-1}(\mathcal{L}_P(\alpha \wedge \beta) - \mathcal{L}_P(\alpha) \wedge \beta) - \alpha \wedge \mathcal{L}_P \beta, \qquad \alpha \in \Omega^k(\mathcal{G}), \beta \in \Omega^l(\mathcal{G}).$$

Here $\mathcal{L}_P : \Omega^n(\mathcal{G}) \to \Omega^{n-1}(\mathcal{G})$ is defined by $\mathcal{L}_P = \iota_P \circ d - d \circ \iota_P$.

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- Are multiplicative 1-forms closed w.r.t $[\cdot, \cdot]_P$?
- Are all multiplicative forms closed w.r.t the Koszul bracket?

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Obstacle: Multiplicative forms are not closed under the wedge product.

Define $\Omega^\bullet_{\mathrm{emult}}(\mathcal{G}) := \{s^*\gamma - t^*\gamma; \gamma \in \Omega^\bullet(M)\}$.

Proposition

For a Poisson groupoid $(\mathcal{G}, P)$, w.r.t. the Koszul bracket $[\cdot,\cdot]_P$, we have that

(1) $\Omega^\bullet_{\mathrm{mult}}(\mathcal{G}) \subset \Omega^\bullet(\mathcal{G})$ *is a graded Lie subalgebra;*

(2) $\Omega^\bullet_{\mathrm{emult}}(\mathcal{G}) \subset \Omega^\bullet_{\mathrm{mult}}(\mathcal{G})$ *is an ideal;*

(3) *The map*

$$P^\sharp : (\Omega^\bullet_{\mathrm{mult}}(\mathcal{G}), [\cdot,\cdot]_P) \to (\mathfrak{X}^\bullet_{\mathrm{mult}}(\mathcal{G}), [\cdot,\cdot])$$

is a graded Lie algebra homomorphism.

Theorem (Chen-L-Liu)

For a Poisson groupoid $(\mathcal{G}, P)$, we have a natural strict graded Lie 2-algebra

$$\Omega^\bullet(M) \xrightarrow{e} (\Omega^\bullet_{\mathrm{mult}}(\mathcal{G}), [\cdot,\cdot]_P), \qquad \gamma \mapsto s^*\gamma - t^*\gamma,$$

where the action is determined by

$$s^*(\Theta \triangleright \gamma) = [\Theta, s^*\gamma]_P, \qquad \forall \Theta \in \Omega^\bullet_{\mathrm{mult}}(\mathcal{G}), \gamma \in \Omega^\bullet(M).$$

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Moreover, $(P^\sharp, p^\sharp)$ is a graded Lie 2-algebra homomorphism:

$$\begin{array}{ccc} \Omega^\bullet(M) & \xrightarrow{p^\sharp} & \Gamma(\wedge^\bullet A) \\ \downarrow e & & \downarrow T \\ \Omega^\bullet_{\mathrm{mult}}(\mathcal{G}) & \xrightarrow{P^\sharp} & \mathfrak{X}^\bullet_{\mathrm{mult}}(\mathcal{G}) \end{array} .$$

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$$\begin{CD} \Omega^\bullet(M) @>{p^\sharp}>> \Gamma(\wedge^\bullet A) \\ @V{e}VV @VV{T}V \\ \Omega^\bullet_{\mathrm{mult}}(\mathcal{G}) @>{P^\sharp}>> \mathfrak{X}^\bullet_{\mathrm{mult}}(\mathcal{G}) \end{CD} \; .$$

Question: What about quasi-Poisson groupoids? Can we get weak Lie 2-algebras?

Quasi-Poisson case

A quasi-Poisson groupoid is a Lie groupoid $\mathcal{G}$ with $P \in \mathfrak{X}^2_{\mathrm{mult}}(\mathcal{G})$ and $\Phi \in \Gamma(\wedge^3 A)$ s.t.

$$\frac{1}{2}[P, P] = \overrightarrow{\Phi} - \overleftarrow{\Phi}, \qquad [P, \overrightarrow{\Phi}] = 0.$$

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Important formulas:

$$[\Theta_1, [\Theta_2, \Theta_3]_P]_P + c.p. = -\frac{1}{2} L_{[P,P](\Theta_1,\Theta_2)}\Theta_3 + c.p. + d\big([P,P](\Theta_1,\Theta_2,\Theta_3)\big);$$

$$P^\sharp[\Theta_1,\Theta_2]_P - [P^\sharp\Theta_1, P^\sharp\Theta_2] = \frac{1}{2}[P,P](\Theta_1,\Theta_2), \qquad \Theta_i \in \Omega^1(\mathcal{G}).$$

Theorem (Chen-L-Liu)

Let $(\mathcal{G}, P, \Phi)$ be a quasi-Poisson groupoid. Then the triple

$$\Omega^1(M) \xrightarrow{J} \Omega^1_{\mathrm{mult}}(\mathcal{G}), \qquad J(\gamma) := s^*\gamma - t^*\gamma,$$

is a weak Lie 2-algebra, where the bracket on $\Omega^1_{\mathrm{mult}}(\mathcal{G})$ is $[\cdot,\cdot]_P$, the action and 3-bracket

$$\triangleright : \Omega^1_{\mathrm{mult}}(\mathcal{G}) \wedge \Omega^1(M) \to \Omega^1(M), \qquad \textit{and} \qquad [\cdot,\cdot,\cdot]_3 : \wedge^3\Omega^1_{\mathrm{mult}}(\mathcal{G}) \to \Omega^1(M)$$

are determined by

$$\begin{aligned} s^*(\Theta \triangleright \gamma) &= [\Theta, s^*\gamma]_P, \\ s^*[\Theta_1, \Theta_2, \Theta_3]_3 &= L_{\overleftarrow{\Phi}(\Theta_1,\Theta_2,\cdot)}\Theta_3 + c.p. - 2d\overleftarrow{\Phi}(\Theta_1, \Theta_2, \Theta_3). \end{aligned}$$

Proof.

For $\Theta_i \in \Omega^1_{\mathrm{mult}}(\mathcal{G})$,

$$\begin{aligned} & s^*\big(\Theta_1 \triangleright [\Theta_2, \Theta_3, \Theta_4]_3 + c.p. - \big([[\Theta_1, \Theta_2]_P, \Theta_3, \Theta_4]_3 + c.p.\big)\big) \\ = \quad & \iota_{[P, \overleftarrow{\Phi}](\Theta_1, \Theta_2, \Theta_3, \cdot)} d\Theta_4 + c.p. + d[P, \overleftarrow{\Phi}](\Theta_1, \Theta_2, \Theta_3, \Theta_4). \end{aligned}$$

■

Theorem (Chen-L-Liu)

Let $(\mathcal{G}, P, \Phi)$ be a quasi-Poisson groupoid. Then the following statements are true:

(a) *The triple $\Omega^\bullet(M) \xrightarrow{J} \Omega^\bullet_{\mathrm{mult}}(\mathcal{G})$ is a graded Lie 2-algebra, where the bracket on $\Omega^\bullet_{\mathrm{mult}}(\mathcal{G})$ is $[\cdot,\cdot]_P$, the action $\triangleright : \Omega^p_{\mathrm{mult}}(\mathcal{G}) \times \Omega^q(M) \to \Omega^{p+q-1}(M)$ and the 3-bracket $[\cdot,\cdot,\cdot]_3 : \Omega^p_{\mathrm{mult}}(\mathcal{G}) \wedge \Omega^q_{\mathrm{mult}}(\mathcal{G}) \wedge \Omega^s_{\mathrm{mult}}(\mathcal{G}) \to \Omega^{p+q+s-2}(M)$ are*

$$\begin{aligned} s^*(\Theta \triangleright \gamma) &= [\Theta, s^*\gamma]_P, \\ s^*[\Theta_1, \Theta_2, \Theta_3]_3 &= d\iota_{\iota_{\iota_{\overleftarrow{\Phi}}\Theta_1}\Theta_2}\Theta_3 + \big(\iota_{\iota_{\iota_{\overleftarrow{\Phi}}\Theta_1}\Theta_2} d\Theta_3 + c.p.\big). \end{aligned}$$

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(b) *There is a weak morphism of graded Lie 2-algebras*

$$\begin{array}{ccc} \Omega^\bullet(M) & \xrightarrow{\wedge^\bullet p^\sharp} & \Gamma(\wedge^\bullet A) \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle T} \\ \Omega^\bullet_{\mathrm{mult}}(\mathcal{G}) & \xrightarrow{\wedge^\bullet P^\sharp} & \mathfrak{X}^\bullet_{\mathrm{mult}}(\mathcal{G}) \end{array},$$

where $\nu : \Omega^p_{\mathrm{mult}}(\mathcal{G}) \wedge \Omega^q_{\mathrm{mult}}(\mathcal{G}) \to \Gamma(\wedge^{p+q-1} A)$ is defined by

$$\nu(\Theta_1, \Theta_2) = (\mathrm{id} \otimes \wedge^{p+q-2} p^\sharp)\big(\iota_\Phi(\theta_1 \wedge \theta_2)\big).$$

For a tensor field $T \in \mathcal{T}^{k,l}(\mathcal{G})$ on $\mathcal{G}$ and $\Theta \in \Omega^p(\mathcal{G})$, define $\iota_T \Theta \in \mathcal{T}^{k-1,l+p-1}(\mathcal{G})$:

$$\iota_T \Theta = \iota_{X_1 \wedge \cdots \wedge X_k \otimes \beta} \Theta := \sum_i (-1)^{k-i+1} X_1 \wedge \cdots \widehat{X_i} \wedge X_k \otimes (\beta \wedge \iota_{X_i} \Theta). \tag{1}$$

Lemma

(a) *For all* $T \in \mathcal{T}^{k,l}_{\text{mult}}(\mathcal{G})$ *and* $\Theta \in \Omega^p_{\text{mult}}(\mathcal{G})$, *we have* $\iota_T \Theta \in \mathcal{T}^{k-1,l+p-1}_{\text{mult}}(\mathcal{G})$;

(b) *For* $u \in \Gamma(\wedge^k A), \gamma \in \Omega^l(M)$ *and* $\Theta \in \Omega^p_{\text{mult}}(\mathcal{G})$, *we have*

$$\iota_{\overleftarrow{u} \otimes s^* \gamma} \Theta = \overleftarrow{v} \otimes s^* \mu,$$

for some $v \in \Gamma(\wedge^{k-1} A)$ *and* $\mu \in \Omega^{l+p-1}(M)$.

Corollary

If (G, P, Φ) *with* $\Phi \in \wedge^3\mathfrak{g}$ *is a quasi-Poisson Lie group, then*

- $(\Omega^1_{\mathrm{mult}}(G), [\cdot,\cdot]_P)$ *is a Lie algebra, although* $\Omega^1(G)$ *is not;*
- $(P^\sharp, 0, \nu)$ *is a weak homomorphism between two strict Lie 2-algebras:*

$$\begin{array}{ccc} 0 & \xrightarrow{\;0\;} & \mathfrak{g} \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle T} \\ \Omega^1_{\mathrm{mult}}(G) & \xrightarrow{\;P^\sharp\;} & \mathfrak{X}^1_{\mathrm{mult}}(G) \end{array} ,$$

where $\nu : \wedge^2\Omega^1_{\mathrm{mult}}(G) \to \mathfrak{g}$ *is given by*

$$\nu(\Theta_1, \Theta_2) = -\Phi(\theta_1, \theta_2), \qquad \theta_i = \mathrm{pr}_{\mathfrak{g}^*}\Theta_i.$$

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More examples: Action Lie groupoids

Thanks for your attention!