# Geometric quantization of Lagrangian torus fibrations

and: a fact about integral-integral affine geometry

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Motivation: "invariance of polarization" phenomenon in geometric quantization.

affine maps $\mathbb{R}^n \longrightarrow \mathbb{R}^n$

$$x \longmapsto Ax+b$$

| | $A$ | $b$ |
|---|---|---|
| | $GL(n,\mathbb{R})$ | $\mathbb{R}^n$ |
| integral affine: | $GL(n,\mathbb{Z})$ | $\mathbb{R}^n$ |
| integral-integral affine: | $GL(n,\mathbb{Z})$ | $\mathbb{Z}^n$ |

$B$ <u>integral-integral affine mfld</u>:

$:\Longleftrightarrow$ mfld $+$ maximal $\underbrace{\text{integral-integral affine atlas}}_{\text{transition maps are locally integral-integral affine}}$.

$\Longrightarrow$

- flat torsion free connection on $TB$ $\Big\}$ affine
- integral lattice $\Lambda \subset TB$
  period lattice $\Lambda^* \subset T^*B$
  $\Longrightarrow$ measure on $B$ $\Bigg\}$ integral affine
- integral points $B_{\mathbb{Z}} \subset B$ $\Big\}$ integral-integral affine

<u>Fact</u>:

$$B \text{ cpct} \implies |B_{\mathbb{Z}}| = \mathrm{vol}(B)$$

Examples: $B = \mathbb{R}^n / \Gamma$, $\Gamma \curvearrowright \mathbb{R}^n$

| $B$ | $n$ | $\Gamma$ generated by ... |
|---|---|---|
| torus | $n \geq 1$ | $x_j \mapsto x_j + 1$ for $j = 1, \dots, n$ |
| Klein bottle | $n = 2$ | $x_1 \mapsto x_1 + 1$, $(x_1, x_2) \mapsto (-x_1, x_2 + 1)$ |
| Kodaira-Thurston-like | $n \geq 3$ | $(x_1, x_2, x_3) \mapsto (x_1 + x_2, x_2, x_3 + 1)$ <br> $x_j \mapsto x_j + 1$ for $j \neq 3$ |

Question:

$B$ integral (integral) affine, cpct $\overset{?}{\Longrightarrow}$ $B \cong \mathbb{R}^n / \Gamma$

Proper Lagrangian fibration :

$$\begin{array}{c} (M, \omega) \\ \pi \downarrow \\ B \end{array}$$

Arnold - Liouville charts :

semilocal coordinates:

$$\begin{array}{ccc} M \supset \pi^{-1}(U) & (x_1, \dots, x_n, t_1, \dots, t_n) & t_j \mod 1 \\ \downarrow & \downarrow & \omega = \sum_{j=1}^{n} dx_j \wedge dt_j \\ B \supset U & (x_1, \dots, x_n) & \end{array}$$

$\Rightarrow$ integral affine str on $B$

Prequantize:

$$\begin{array}{c} (L, \langle\,\rangle, \nabla) \\ \downarrow \\ (M, \omega) \\ \downarrow \\ B \end{array} \qquad \text{curvature } \nabla = \omega$$

Enhanced Arnold Liouville charts:

Arnold-Liouville charts + local trivializations of $L$

s.t. $\nabla = d + i \sum_j x_j \, dt_j$

$\Rightarrow$ integral-integral affine str. on $B$

$$\begin{array}{c} M_0 := T^*B/\Lambda^* \\ \downarrow \\ B \end{array}$$

Lagrangian torus fibration

$$\begin{array}{c} M^\vee := TB/\Lambda \\ \pi \downarrow \\ B \end{array}$$

semi-local coordinates:

$$\underbrace{x_1, \dots, x_n}_{\substack{\text{integral-integral} \\ \text{local coordinates} \\ \text{on } B}}, \quad \underbrace{y_1, \dots, y_n}_{\substack{\text{mod } 1 \\ \text{adapted fibre coordinates}}}$$

Local torus $\mathbb{T}^n$ action on $M^\vee$

up to $\mathrm{Aut}(\mathbb{T}^n)$ on chart overlaps

$\alpha \in \Omega^k(W)$, $\quad W = \pi^{-1}(U) \underset{\text{open}}{\subset} M^\nu$

"$\alpha$ is $\mathbb{T}^n$-invariant"

$:\Longleftrightarrow$ locally $\alpha = g^*\alpha \quad \forall g \in \mathbb{T}^n$

is well defined.

$\Rightarrow$ Complex $(\Omega^\bullet_{\text{inv}}(M^\nu), d)$

$\Rightarrow$ Cohomology $H^\bullet_{\text{inv}}(M^\nu)$

$$\Omega^\bullet_{\text{inv}}(M^\nu) \xrightarrow{i} \Omega^\bullet(M^\nu)$$

$\Rightarrow$ $i_*$: $H^\bullet_{\text{inv}}(M^\nu) \longrightarrow H^\bullet(M^\nu)$

<u>Lemma</u> $i_*$ is an isomorphism.

pf for global $\mathbb{T}^n$ action:

averaging $\quad \bar{\alpha} := \int_{g \in \mathbb{T}^n} (g^* \alpha) \, dg$

$d\bar{\alpha} = \overline{d\alpha}$.

$\mathbb{T}^n$ Connected $\Rightarrow \overset{\forall g}{[g^*\alpha] = [\alpha]} \Rightarrow [\bar{\alpha}] = [\alpha]$

$\therefore$ $H^i_{inv} \to H^i$ is onto.

If $\alpha$ is invariant and $\alpha = d\beta$.

Then also $\alpha = d\bar{\beta}$.

$\therefore$ $H^i_{inv} \to H^i$ is one-to-one.

pf for local $\mathbb{T}^n$-action: induction on size of covering by open sets w/ $\mathbb{T}^n$ actions.

$l = 1$: Case of global $T^n$ action.

$l \rightsquigarrow l+1$: Mayer Vietoris + five lemma:

$$\begin{tikzcd}
\cdots \arrow[r] & H^k_{inv}\left(\bigcup_{i=1}^{l+1} W_i\right) \arrow[r] \arrow[d, "i_*"] & H^k_{inv}\left(\bigcup_{i=1}^{l} W_i\right) \oplus H^k_{inv}(W_{l+1}) \arrow[r] \arrow[d, "i_* \oplus i_*"] & H^k_{inv}\left(\left(\bigcup_{i=1}^{l} W_i\right) \cap W_{l+1}\right) \arrow[r] \arrow[d, "i_*"] & \cdots \\
\cdots \arrow[r] & H^k\left(\bigcup_{i=1}^{l+1} W_i\right) \arrow[r] & H^k\left(\bigcup_{i=1}^{l} W_i\right) \oplus H^k(W_{l+1}) \arrow[r] & H^k\left(\left(\bigcup_{i=1}^{l} W_i\right) \cap W_{l+1}\right) \arrow[r] & \cdots
\end{tikzcd}$$

$\blacksquare$

Note. $av_* \circ i_* = $ Identity on $H^\bullet$

& $i_*$ is an isomorphism

$\Rightarrow$ also $i_* \circ av_* = I$

$B$ integral affine, $\Lambda \subset TB$ integral lattice.

$M^\vee = TB/\Lambda$. $\quad x_1, \dots, x_n, \underbrace{y_1, \dots, y_n}_{\text{mod } 1}$ semi-local adapted coordinates.

$\downarrow$

$B$

$Z_0 := \{y_j = 0\}$ zero section.

Assume $B$ is oriented.

<u>Thm</u>.

- $dy_1 \wedge \dots \wedge dy_n$ is independent of the choice of adapted coordinates
- $[dy_1 \wedge \dots \wedge dy_n] = P.D.(Z_0)$

• $[dy_1 \wedge \dots \wedge dy_n] = P.D.(Z_0)$

pf. Take any coh. class $[\alpha] \in H^n$.

WLG $\alpha$ is invt.

Locally $\alpha = a(x)\, dx_1 \wedge \dots \wedge dx_n$.

$$\int_{Z_0} \alpha = \int_{Z_0} a(x)\, dx_1 \wedge \dots \wedge dx_n$$

$$\underset{\text{Fubini}}{=} \int_{M^r} a(x)\, dx_1 \wedge \dots \wedge dx_n \wedge dy_1 \wedge \dots \wedge dy_n$$

$$= \int_{M^r} \alpha \wedge (dy_1 \wedge \dots \wedge dy_n).$$

assume $B$ is integral - integral affine.

$Z_0 := \{y_j = 0\} \subset M^\vee$ zero section

$D := \{y_j = x_j \mod 1\} \subset M^\vee$ diagonal section.

$B_{\mathbb{Z}} = \{x \mid x_j = 0 \mod 1\}$

$|Z_0 \cap D| = |B_{\mathbb{Z}}|$

pf that $|B_{\mathbb{Z}}| = \mathrm{vol}(B)$:

$$|B_{\mathbb{Z}}| = |D \cap Z_0| = \int_D [dy_1 \wedge \cdots \wedge dy_n]$$

$$= \int_B \underbrace{\sigma^* dy_1 \wedge \cdots \wedge dy_n}_{= dx_1 \wedge \cdots \wedge dx_n}$$

$\sigma : x_j \mapsto (x_j, x_j \mod 1)$

$$= \mathrm{vol}(B).$$

# Bohr-Sommerfeld quantization

$$\begin{array}{c} L \\ \downarrow \\ M \\ \pi \downarrow \\ B \end{array}$$

$x \in B$ Bohr-Sommerfeld

$:\Longleftrightarrow$ $\left(L|_{\pi^{-1}(x)}, \nabla\right)$ is trivial

$$\boxed{\Longleftrightarrow \quad x \in B_{\mathbb{Z}}.}$$

The Hilbert space: $\mathcal{H}_{BS} := \bigoplus_{\text{Bohr Sommerfeld}} \mathbb{C}$.

## Dirac-Dolbeault quantization:

The Hilbert space: $\mathcal{H}$ s.t.

$$\dim \mathcal{H} = RR(M,\omega) := \int_M e^{\omega} \, \mathrm{Todd}(TM, J).$$

cptble a.c. str.

The RHS:

$$e^{\omega} = 1 + \omega + \frac{\omega^2}{2!} + \dots$$

$$\int_{M^n} e^{\omega} = \int_M \frac{\omega^n}{n!}$$ Liouville volume

$$\mathrm{Todd}(x) := \prod_j \frac{x_j}{1-e^{-x_j}} \rightsquigarrow \mathrm{Todd}: \mathrm{Lie}(u(n)) \to \mathbb{R}$$

$$\underset{\text{Chern Weil}}{\rightsquigarrow} \mathrm{Todd}: \left\{ \begin{array}{c} \text{cplx v. bundles} \\ \text{over } M \end{array} \right\} \longrightarrow H^{\bullet}(M, \mathbb{R})$$

The LHS: e.g., can take

$$\mathcal{H} = \Gamma_{hol}(L) \quad \left\{ \text{ if } \begin{array}{l} J \text{ is Kähler} \\ \& \\ H^{\geq 1}(\mathcal{O}_L) = 0 \end{array} \right.$$

or $$\mathcal{H} = \Sigma (-1)^i H^i(\mathcal{O}_L) \quad \left\{ \text{ if } J \text{ is Kähler} \right.$$

or $$\mathcal{H} = \ker \bar{\partial}_L - \operatorname{coker} \bar{\partial}_L \quad \left\{ \text{ if } J \text{ is almost cplx} \right.$$

Theorem:

$$RR(M, \omega) = |B_{\mathbb{Z}}|$$

Bohr sommerfeld set.

interpretation: "invariance of polarization"

$$\begin{array}{c} L \\ \downarrow \\ (M, \omega) \\ \pi \downarrow \\ B \end{array}$$

## Proof

$V := \ker \pi_* \subset TM$

$$0 \to V \to TM \to \pi^* TB \to 0$$

vector bundles over $M$

$V \overset{\omega}{\cong} \pi^* T^* B$ is flat

$V \otimes \mathbb{C} \cong (TM, J)$ via $u + iv \longmapsto u + Jv$

$u, v \in V_y \qquad y \in M$

$$\Rightarrow \quad (TM, J) \cong \pi^* \big( \underbrace{T^* B \otimes \mathbb{C}}_{\text{flat}} \big)$$

$\Rightarrow$ $(TM, J)$ is flat. $\Rightarrow$ $\mathrm{Todd}(TM, J) = 1$

$$\Rightarrow \quad RR(M,\omega) = \int_M e^{\omega} \underbrace{1}_{\mathrm{Todd}}$$

$$= \text{Liouville volume of } M$$

$$\overset{\text{Fubini}}{=} \text{integral affine volume of } B$$

$$= |B_{\mathbb{Z}}|$$

By the fact from integral-integral geometry

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