Normal forms for Logarithmic connections

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Fuchsian singularities

Linear ODE on $\mathbb{A}^1$ with Fuchsian singularity at the origin:

$$z\frac{ds}{dz} = A(z)s,$$

where $A(z)$ is a holomorphic $n \times n$ matrix.

Gauge transformations: $s = g^{-1}t$, for $g(z) \in GL(n)$. Then t satisfies

$$z\frac{dt}{dz} = B(z)t,$$

where $B(z) = gAg^{-1} + zg'(z)g^{-1}$.

Important names: Hukuhara, Turrittin, Levelt, Gantmacher, Babbitt and Varadarajan, ...

Fuchsian singularities

Fundamental Theorem

After a holomorphic gauge transformation

$$A(z) = S + \sum_{i \geq 0} N_i z^i,$$

where

- S is diagonal,
- N_i is nilpotent and satisfies $[S, N_i] = iN_i$,

- $S + N_0$ is the linear approximation to $A(z)$,
- The other N_i are the resonant correction terms.

Levelt: This defines a Jordan decomposition.

Implications for geometry

Fix $r = x + iy + n \in End(\mathbb{C}^n)$. This determines

1. a parabolic $P_x \subseteq GL_n(\mathbb{C})$,
2. a Levi $L_x = C(x) \subset P_x$
3. The conjugacy class of $\exp(2\pi i r)$: $\mathcal{C} \subset L_x$.

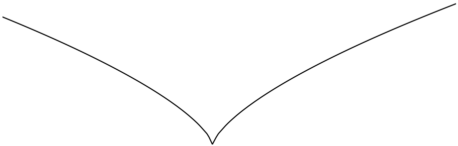
- Boalch (also Deligne and Simpson): The moduli space of Fuchsian equations with $A(0)$ in the adjoint orbit of r isomorphic to

$$(X_r := \{(M, P) \mid P \sim P_x, \quad M \in P, \quad \pi(M) \in \mathcal{C}\})/GL_n(\mathbb{C})$$

- Boalch: X_r is a quasi-Hamiltonian G-space.
- Safronov: $X_r/G \to G/G$ is a 1-shifted Lagrangian.

Cusp Singularity $x^2 = y^3$

Consider the singular curve $D \subseteq \mathbb{C}^2$ defined by $x^2 = y^3$:



Goal: study flat connections on $\mathbb{C}^2$ with logarithmic singularities along D.

Cusp Singularity $f = x^2 - y^3$

1. Logarithmic flat connection:

$$\nabla = d + \frac{A}{6f}df + \frac{B}{6f}(3xdy - 2ydx),$$

where $A(x,y), B(x,y) \in End(\mathbb{C}^n)$ holomorphic and satisfy

$$V(A) - E(B) + B + [A, B] = 0,$$

where

$$E = 3x\partial_x + 2y\partial_y, \qquad V = 3y^2\partial_x + 2x\partial_y.$$

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2. These are representations of the logarithmic tangent bundle

$$T_{\mathbb{C}^2}(-\log D) = \text{Vector fields on } \mathbb{C}^2 \text{ tangent to } D$$

Lie algebroid with basis E and V, and where $[E,V] = V$.

Saito Free divisors

- We know that $T_X(-\log D)$ is locally free when $D \subset X$ is a smooth hypersurface.
- This may fail when D is singular. This general situation was studied by K. Saito.

Theorem (Saito's Criterion)

A hypersurface $D \subset \mathbb{C}^n$ is a free divisor at a point x if and only if there exist n vector fields $X_1, ..., X_n \in T_{\mathbb{C}^n}(-\log D)$ such that

$$X_1 \wedge ... \wedge X_n = f \cdot vol,$$

with f a reduced equation for D near x.

Saito: Plane curves are always free $\implies x^2 = y^3$ is free.

Cusp Singularity $f = x^2 - y^3$

Theorem (B.)

After a holomorphic gauge transformation, the connection has the form

$$\nabla = d + \frac{(S+N)}{6f} df + \frac{B}{6f}(3xdy - 2ydx),$$

where S is constant diagonal, $N(x,y)$ is a holomorphic nilpotent matrix, $B(x,y)$ holomorphic, and

$$\begin{aligned} E(B) &= B + [S,B] \\ E(N) &= [S,N] \\ V(N) &= [B,N]. \end{aligned}$$

$E(B) = B + [S, B]$

- The vector field $E = 3x\partial_x + 2y\partial_y$ defines a grading of coordinates: $|x| = 3, |y| = 2$.
- B_k, the weight k component of B, is an eigenvector for ad_S:

$$[S, B_k] = (k-1)B_k.$$

- Therefore, $E(B) = B + [S, B]$ defines a finite dimensional vector space.

Implications for geometry

Fix S. Then flat connections are parametrized by (B, N). The first two equations define a finite dimensional affine space

$$W_S = \{(B, N) \mid E(B) = B + [S, B], \quad E(N) = [S, N]\}.$$

The last equation defines an algebraic subvariety

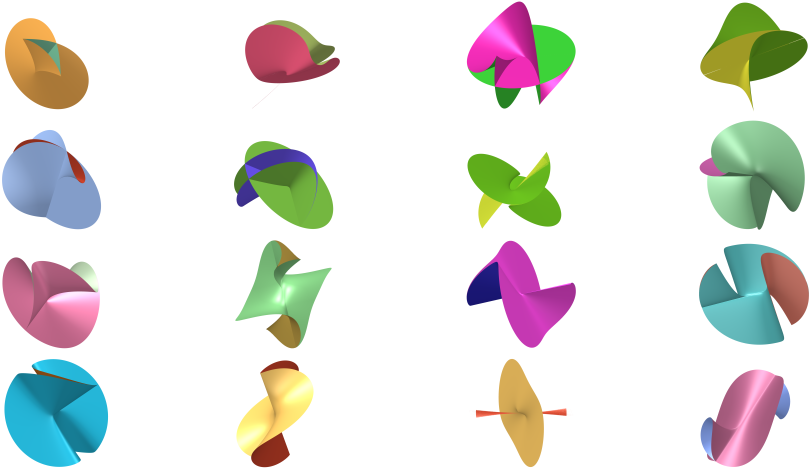
$$X_S = \{(B, N) \in W_S \mid V(N) = [B, N], \quad N \text{ is nilpotent}\} \subset W_S.$$

Corollary (B.)

The moduli space of logarithmic flat connections with poles along the cusp and 'semisimple residue' conjugate to S has the structure of an algebraic quotient stack

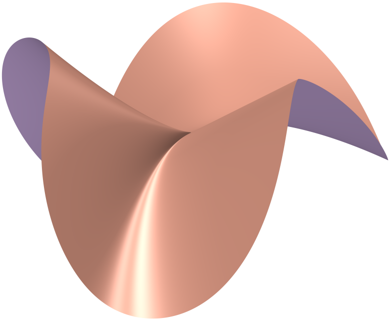
$$[X_S / Aut(S)].$$

Sekiguchi's 17 free divisors



Sekiguchi's 17 free divisors

- Sekiguchi gave a classification of weighted homogeneous free divisors in $\mathbb{C}^3$. There are 17 examples.
- One of the examples is the vanishing of $F = xy^4 + y^3z + z^3$:



Sekiguchi's free divisor $F = xy^4 + y^3z + z^3$

$T_{\mathbb{C}^3}(-\log D)$ has the following basis

$$\begin{aligned} E &= x\partial_x + 2y\partial_y + 3z\partial_z \\ V &= 2y\partial_x + (-24xy + 2z)\partial_y + (-2y^2 - 32xz)\partial_z \\ W &= 3z\partial_x - 9y^2\partial_y - 12yz\partial_z, \end{aligned}$$

with brackets

$$[E, V] = V, \quad [E, W] = 2W, \quad [V, W] = 24zE + 6yV - 40xW.$$

Sekiguchi's free divisor $F = xy^4 + y^3z + z^3$

$T_{\mathbb{C}^3}(-\log D)$ has a basis E, V, W. Dual basis α, β, γ of log 1-forms.

A flat logarithmic connection is given by

$$\nabla = d + A\alpha + B\beta + C\gamma,$$

where $A, B, C \in End(\mathbb{C}^n)$ are holomorphic functions on $\mathbb{C}^3$ that satisfy

$$\begin{aligned} &E(B) - V(A) - B + [B, A] = 0 \\ &E(C) - W(A) - 2C + [C, A] = 0 \\ &V(C) - W(B) - 24zA - 6yB + 40xC + [C, B] = 0. \end{aligned}$$

Sekiguchi's free divisor $F = xy^4 + y^3z + z^3$

Theorem (B.)

After a holomorphic gauge transformation, the connection has the form

$$\nabla = d + (S + N)\alpha + (B - \frac{32}{3}xN)\beta + (C - 4yN)\gamma,$$

where S is constant diagonal, N is holomorphic and nilpotent, B and C are holomorphic, and

$$\begin{aligned} E(B) &= B + [S, B] \\ E(C) &= 2C + [S, C] \\ E(N) &= [S, N] \\ V(N) &= [B, N] \\ W(N) &= [C, N] \\ V(C) - W(B) &= 24zS + 6yB - 40xC - [C, B]. \end{aligned}$$

Sekiguchi's free divisor $F = xy^4 + y^3z + z^3$

Corollary (B.)

The moduli space of logarithmic flat connections with poles along Sekiguchi's hypersurface and 'semisimple residue' conjugate to S has the structure of an algebraic quotient stack

$$[X_S/Aut(S)].$$

Corollary (B.)

$T_{\mathbb{C}^3}(-\log D)$ is not an action algebroid

$$T_{\mathbb{C}^3}(-\log D) \ncong \mathfrak{g} \ltimes \mathbb{C}^3.$$

Plan for the talk

- Discuss the proof of the normal form theorem in the case of the cusp curve.
- Strategy: integrate $T_{\mathbb{C}^2}(-\log D)$ to a Lie groupoid $\Pi(X, D)$ and study it's representation theory.
- Relies on the theory of 'homogeneous Lie groupoids'.
- Two main results: Linearization and Jordan decomposition.

Cusp Singularity $f = x^2 - y^3$

1. Logarithmic tangent bundle $T_{\mathbb{C}^2}(-\log D)$ has basis of sections

$$E = 3x\partial_x + 2y\partial_y, \qquad V = 3y^2\partial_x + 2x\partial_y,$$

   which satisfy $[E, V] = V$. Therefore,

$$T_{\mathbb{C}^2}(-\log D) \cong \mathfrak{g}_A \ltimes \mathbb{C}^2,$$

   where $\mathfrak{g}_A$ is the non-abelian Lie algebra of dimension 2.

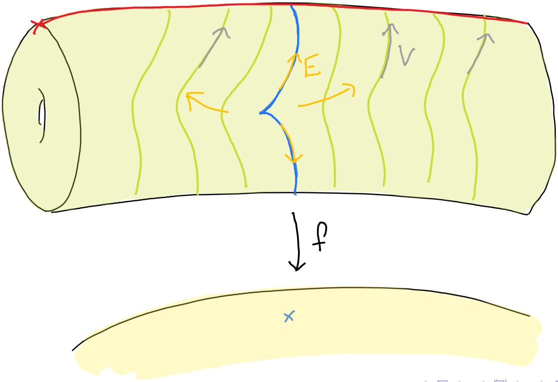
2. Source-simply connected integration is the **twisted fundamental groupoid** $\Pi(\mathbb{C}^2, D)$: 'homotopy classes of paths in $\mathbb{C}^2$ tangent to D'.

3. Lie's second theorem:

$$\left\{ \begin{array}{c} \text{Flat connections } \nabla \text{ with} \\ \text{logarithmic poles along } D \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Groupoid representations} \\ \phi : \Pi(\mathbb{C}^2, D) \to GL_n(\mathbb{C}) \end{array} \right\}$$

Twisted fundamental groupoid

- $T_{\mathbb{C}^2}(-\log D) \cong \mathfrak{g}_A \ltimes \mathbb{C}^2$, but $\Pi(\mathbb{C}^2, D) \ncong G_A \ltimes \mathbb{C}^2$, for $G_A = \mathbb{C} \ltimes \mathbb{C}$, since the vector field V is not complete.
- $f = x^2 - y^3 : \mathbb{C}^2 \to \mathbb{C}$ fibration of $\mathbb{C}^2$ by once punctured elliptic curves.

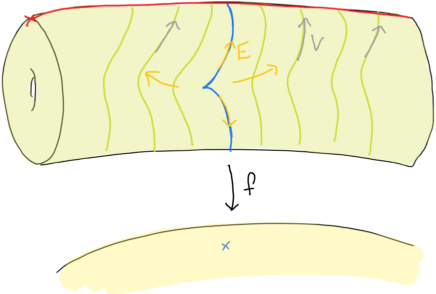


Twisted fundamental groupoid

- Fibrewise compactification $X = \overline{\mathbb{C}^2}$, $\overline{D} \subset X$.
- Then $\Pi(X, \overline{D}) \cong G_A \ltimes X$.
- But $\Pi(X, \overline{D})|_{\mathbb{C}^2} \ncong \Pi(\mathbb{C}^2, D)$ since

$$\pi_1(X \setminus \overline{D}) \cong \mathbb{Z} \ltimes \mathbb{Z}^2, \qquad \pi_1(\mathbb{C}^2 \setminus D) \cong \mathbb{Z} \ltimes F_2$$

So take the Weinstein groupoid of $\Pi(X, \overline{D})|_{\mathbb{C}^2}$.



Subgroupoids

- There is a sub-action groupoid

$$r : \mathbb{C} \ltimes \mathbb{C}^2 \to \Pi(\mathbb{C}^2, D)$$

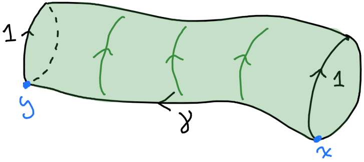
$$(\lambda, x, y)$$

$$(e^{3\lambda}x, e^{2\lambda}y) \longleftarrow (x, y)$$

- Subgroupoid of isotropy elements

$$i : \mathbb{Z} \times \mathbb{C}^2 \to \mathbb{C} \ltimes \mathbb{C}^2 \to \Pi(\mathbb{C}^2, D).$$

This subgroupoid is *central*.



Restricting representations

We can restrict representations along groupoid homomorphisms

$$r^* : Rep(\Pi(\mathbb{C}^2, D)) \to Rep(\mathbb{C} \ltimes \mathbb{C}^2),$$

$$\phi : \Pi(\mathbb{C}^2, D) \to GL_n(\mathbb{C}) \ \mapsto \ \phi \circ r : \mathbb{C} \ltimes \mathbb{C}^2 \to GL_n(\mathbb{C})$$

Monodromy automorphism

Given a representation

$$\phi : \Pi(\mathbb{C}^2, D) \to GL_n(\mathbb{C})$$

Restrict ϕ along the isotropy subgroupoid $i : \mathbb{Z} \times \mathbb{C}^2 \to \Pi(\mathbb{C}^2, D)$:

$$M(x, y) := \iota^* \phi(1, x, y)$$

Lemma: Because $\mathbb{Z} \times \mathbb{C}^2$ is central, M is an automorphism of ϕ.

First main result: Linearization

Goal: Given

$$\phi \in Rep(\Pi(\mathbb{C}^2, D)),$$

we want to linearize

$$r^*\phi \in Rep(\mathbb{C} \ltimes \mathbb{C}^2).$$

Linearization: Definition

Morphisms of groupoids

$$\mathbb{C} \underset{p}{\overset{\iota}{\rightleftarrows}} \mathbb{C} \ltimes \mathbb{C}^2$$

induce functors

$$Rep(\mathbb{C}) \underset{p^*}{\overset{\iota^*}{\leftrightarrows}} Rep(\mathbb{C} \ltimes \mathbb{C}^2)$$

Linear approximation functor

$$L := p^* \iota^* : Rep(\mathbb{C} \ltimes \mathbb{C}^2) \to Rep(\mathbb{C} \ltimes \mathbb{C}^2).$$

Given a representation ψ, a linearization is an isomorphism $g : \psi \to L(\psi)$.

Linearization: Obstruction

- The linear approximation of ψ has the form

$$L(\psi)(\lambda, x, y) = \exp(\lambda A),$$

  for $A \in End(\mathbb{C}^n)$.
- The monodromy of $L(\psi)$ is constant $L(M)(x, y) = \exp(2\pi i A)$.
- Hence, if $g : \psi \to L(\psi)$ is a linearization, the monodromy of ψ:

$$M(x, y) = g(x, y) \exp(2\pi i A) g(x, y)^{-1}$$

  lies in a **unique** conjugacy class of $GL_n(\mathbb{C})$.

Linearization

Theorem (B.)

A representation $\psi : \mathbb{C} \ltimes \mathbb{C}^2 \to GL_n(\mathbb{C})$ can be linearized if and only if the monodromy M lies in a unique conjugacy class.

Corollary

A representation can be linearized if it has semisimple monodromy.

proof

- Given representation $\psi : \mathbb{C} \ltimes \mathbb{C}^2 \to GL_n(\mathbb{C})$.
- Let $L(\psi)$ be the linear approximation.

The goal is to find an isomorphism between ψ and $L(\psi)$:

$$g : \mathbb{C}^2 \to GL_n(\mathbb{C})$$

such that

$$g(e^{3\lambda}x, e^{2\lambda}y)L(\psi)(\lambda, x, y) = \psi(\lambda, x, y)g(x, y).$$

proof

Define an action of $\mathbb{C}$ on $(\mathbb{C}^2 \times GL_n(\mathbb{C})) \to \mathbb{C}^2$:

$$\lambda * (x, y, g) = (e^{3\lambda}x, e^{2\lambda}y, \psi(\lambda, x, y)gL(\psi)(\lambda, x, y)^{-1}).$$

Reformulation: we are looking for an invariant section.

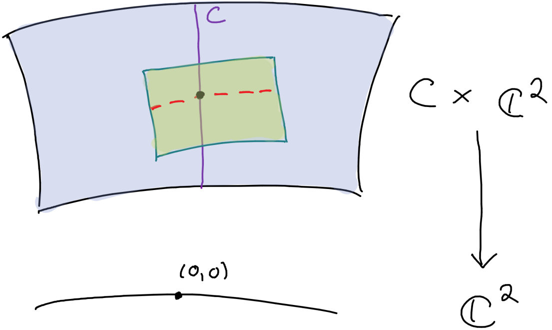
1. Conjugate ψ so that $M = L(\psi)(2\pi i)$ is constant.
2. $C = C(M)$ centraliser. Then $\mathbb{C}$ acts on $\mathbb{C}^2 \times C$.
3. Action descends to $\mathbb{C}^*$.

proof

4 Linearize at the fixed point $(0,0,I) \in \mathbb{C}^2 \times C$. The result is a linear action of $\mathbb{C}^*$ on $T_{(0,0,I)}(\mathbb{C}^2 \times C)$, and a short exact sequence of representations:

$$0 \to T_I(C) \to T_{(0,0,I)}(\mathbb{C}^2 \times C) \to \mathbb{C}^2 \to 0.$$

5 A splitting of this sequence gives an invariant section.



Recap

Goal: Given

$$\phi \in Rep(\Pi(\mathbb{C}^2, D)),$$

we want to linearize

$$r^*\phi \in Rep(\mathbb{C} \ltimes \mathbb{C}^2).$$

This is obstructed by the unipotent part of the monodromy.

Goal: Deform ϕ in order to make its monodromy semisimple.

Cocycle construction

Given a representation $\phi : \mathcal{G} \to GL_n(\mathbb{C})$, a 1-**cocycle** is a map

$$\sigma : \mathcal{G} \to GL_n(\mathbb{C})$$

such that $\sigma(g)\phi(g)$ defines a new representation.

Proposition

Let K be a Lie group, and let $\pi : \mathcal{G} \to K$ be a homomorphism. Given a representation $\phi : \mathcal{G} \to GL_n(\mathbb{C})$, suppose that we have a homomorphism

$$c : K \to Aut(\phi).$$

Then there is a 1-cocycle σ defined by

$$\sigma(g) := c(\pi(g))|_{t(g)}.$$

More about the twisted fundamental groupoid

- Recall that D is cut out by $f = x^2 - y^3 : \mathbb{C}^2 \to \mathbb{C}$.
- Lemma:

$$\pi := \frac{1}{6}\delta(\log f) = \frac{1}{6}(t^* \log(f) - s^* \log(f)) : \Pi(\mathbb{C}^2, D) \to \mathbb{C}$$

  is a well-defined groupoid morphism.
- $\pi \circ r = p : \mathbb{C} \ltimes \mathbb{C}^2 \to \mathbb{C}$ is the projection map.

What do we have so far:

- Homomorphism $\pi : \Pi(X, D) \to \mathbb{C}$.
- Given a representation ϕ, a monodromy homomorphism

$$M : \mathbb{Z} \to Aut(\phi).$$

This is almost the setting for constructing a cocycle: We need an extension

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{M} & Aut(\phi) \\ \downarrow & \nearrow & \\ \mathbb{C} & & \end{array}$$

Jordan-Chevalley decomposition

- An invertible matrix $M \in GL_n(\mathbb{C})$ has a unique factorization

$$M = SU,$$

where S is diagonalizable, U is unipotent ($(U - I)^k = 0$ for large k) and $SU = US$.

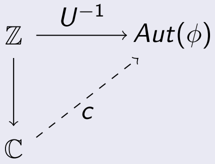
- The logarithm is well-defined and algebraic on unipotent matrices

$$\log(U) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}(U - I)^k.$$

Jordan-Chevalley decomposition

Lemma

- The unipotent part U of the monodromy M defines an automorphism of ϕ:
$$U \in Aut(\phi).$$
- The map
$$c : \mathbb{C} \to Aut(\phi), \qquad z \mapsto \exp(-z \log U)$$
extends $U^{-1} : \mathbb{Z} \to Aut(\phi)$.



Jordan-Chevalley decomposition

Corollary

There is an equivalence of categories

$$Rep(\Pi(\mathbb{C}^2, D)) \cong \mathcal{JC} := \left\{ \begin{array}{l} \text{Representations with} \\ \text{semisimple monodromy} + \\ \text{Unipotent automorphism} \end{array} \right\}$$

$$\phi \mapsto (\phi_s := \sigma\phi, U).$$

This generalizes the Jordan decomposition of Hukuhara, Turrittin and Levelt

proof of normal form

Start with flat connection $\nabla = d + \omega$ with logarithmic singularities.

1. Integrate to representation $\phi \in Rep(\Pi(\mathbb{C}^2, D))$
2. Factorise $\phi \mapsto (\phi_s, U)$.
3. Restrict $\psi = r^*\phi_s \in Rep(\mathbb{C} \ltimes \mathbb{C}^2)$.
4. Linearize $g : \psi \to L(\psi)$.

Then $g * \nabla = d + g\omega g^{-1} + dgg^{-1}$ is in normal form.

Thank You