On 2-shifted lagrangian groupoids and Poisson geometry

(joint work with H. Bursztyn and M. Cueca)

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We will use shifted symplectic geometry to explain several constructions of symplectic groupoids such as:

- The integration of quasi-Poisson spaces
- The integration of Poisson homogeneous spaces.
- Finally, we will mention how to construct more general 2-shifted lagrangian groupoids

Motivation: symplectic reduction

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- (M, ω) hamiltonian G-space
- $(\mathcal{O}, \omega_{KKS}) \hookrightarrow \mathfrak{g}^*$ coadjoint orbit
- if the moment maps are transverse

$$(M, \omega) \xrightarrow{\mu} \mathfrak{g}^* \longleftarrow (\mathcal{O}, \omega_{KKS})$$

Then the fibred product $M \times_{\mathfrak{g}^*} \overline{\mathcal{O}}$ is symplectic mod the G-action

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- quasi-hamiltonian reduction (Alekseev-Malkin-Meinrenken)
- reduction by quasi(/pre)-symplectic groupoid actions (Bursztyn-Crainic, Xu)

How to produce symplectic groupoids using reduction?

This was all about producing symplectic *manifolds* by reduction but what about symplectic *groupoids*?

Reduction techniques for symplectic groupoids

"Multiplicative moment map reduction": the moment maps are groupoid morphisms

- Multiplicative Marsden-Weinstein (Mikami-Weinstein, Fernandes-Ortega-Ratiu):

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There are more things: double symplectic groupoid actions on symplectic groupoids (Stefanini)...

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- Symplectic groupoids of Poisson homogeneous spaces (Bursztyn-Iglesias-Lu)
- Configuration symplectic groupoids of flags (Lu-Mouquin-Yu)

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- We have to extend our concept of moment map to accomodate the "ad hoc" examples that we mentioned
- The required concept is that of (2-shifted) lagrangian structures on Lie groupoid morphisms

Shifted symplectic geometry (PTVV) in differential geometry...

(Following Getzler's slides)

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- m-shifted n-form $\omega_\bullet$ on $K_1 \rightrightarrows K_0$ is $(\omega_i \in \Omega^{m+n-i}(K_i))_{i=0\dots m}$, $\omega_m \in \Omega^n(X_m)$ the leading term

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$\omega_\bullet$ is closed if $D\omega_\bullet = 0$

Infinitesimal form of a closed n-shifted 2-form $\omega_\bullet$

Shift 1:

$$\begin{array}{ccc} \mathrm{Lie}(K_1) & \xrightarrow{\mathtt{a}} & TK_0 \\ \downarrow{\scriptstyle \lambda_\omega} & & \downarrow{\scriptstyle -(\lambda_\omega)^*} \\ T^*K_0 & \xrightarrow[\mathtt{a}^*]{} & \mathrm{Lie}(K_1)^*. \end{array}$$

$$\langle \lambda_\omega(v), x\rangle = \omega_{1|1_p}(v, T1_p(x)) \quad v \in Lie(K_1)_p,\ x \in T_pK_0$$

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Shift 2:

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathrm{Lie}(K_1) & \xrightarrow{\ \mathtt{a}\ } & TK_0 \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \lambda_\omega} & & \Big\downarrow \\ T^*K_0 & \xrightarrow[\ \mathtt{a}^*\ ]{} & \mathrm{Lie}(K_1)^* & \longrightarrow & 0; \end{array}$$

$$\langle \lambda_\omega(v), w\rangle = -\omega_{2|(1_p,1_p)}\big((v, 0_{1_p}), (0_{1_p}, w)\big) + \omega_{2|(1_p,1_p)}\big((0_{1_p}, v), (w, 0_{1_p})\big)$$

$v, w \in \mathrm{Lie}(K_1)_p$

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- 1-shifted symplectic groupoids are quasi(/pre)-symplectic groupoids (Xu, Bursztyn-Crainic-Weinstein-Zhu) (meaning of non-degeneracy equation also explained by del Hoyo-Ortiz)
- 2-shifted symplectic groupoids have surjective anchor

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$$\omega_1 = \frac{1}{12}\langle[\theta^l \wedge \theta^l] \wedge \theta^l\rangle \in \Omega^3(K), \quad \omega_2 = -\frac{1}{2}\langle \mathrm{pr}_1^*\theta^l \wedge \mathrm{pr}_2^*\theta^r\rangle \in \Omega^2(K_2 = K^2)$$

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- λ_ω is contraction with the pairing $\langle \, , \, \rangle$

Moment maps are lagrangian morphisms

- An n-shifted isotropic structure $\eta_\bullet$ on a Lie groupoid morphism

  $$\phi_\bullet : X_\bullet \to K_\bullet$$

  for $(K_\bullet, \omega_\bullet)$ n-shifted symplectic is a (normalised) $n-1$-shifted 2-form which is a primitive for $\phi_\bullet^* \omega_\bullet$:

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- $\eta_\bullet$ is *n-shifted lagrangian* if it induces a quasi-isomorphism

$$(\lambda_\eta \oplus \phi^* \lambda_\omega) : \overbrace{\mathcal{T}_X \oplus \phi^* \mathcal{T}_K[-1]}^{\text{mapping cone of Lie}(\phi)} \to \mathcal{T}_X^*[n-1]$$

$$\partial = \begin{bmatrix} \mathsf{a}_X & 0 \\ \mathcal{T}_\phi & -\mathsf{a}_K \end{bmatrix}$$

2-shifted lagrangian groupoids in a Lie group $K_\bullet$

Concretely:

2-shifted lagrangian groupoids in $K_\bullet$:

$(\phi : X_1 \to K, \eta_\bullet = \eta_0 + \eta_1)$ such that:

- $D\eta_\bullet = \phi_\bullet^* \omega_\bullet$
- $(\mathrm{Lie}(\phi), \mathsf{a}, \eta_1^\flat)$ embeds $\mathrm{Lie}(X_1)$ as a Dirac structure in $\mathfrak{k} \oplus \mathbb{T}_{\eta_0} X_0$

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- lagrangian subgroups of K

Why do we care about these objects?

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Symplectic reduction is given by intersection (fibred products) of lagrangians

Lagrangian "intersections" are symplectic (PTVV): shift 1

- Calaque: hamiltonian G-spaces correspond to 1-shifted lagrangian groupoids of the form

$$(G \times M \rightrightarrows M, \omega) \to (G \times \mathfrak{g}^* \rightrightarrows \mathfrak{g}^*, \Omega_{\mathsf{can}} + 0)$$

- Safronov: q-hamiltonian G-spaces correspond to 1-shifted lagrangian groupoids

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- the intersection of two 1-shifted lagrangian morphisms is their fibred product which is 0-symplectic
- This principle generalises Marsden-Weinstein reduction and quasi-hamiltonian reduction
- (and also the more general reduction by quasi-symplectic groupoid actions)

Lagrangian "intersections" are symplectic (PTVV): shift 2

Another instance of this principle

$(\phi, \eta) : X_1 \to K$, $(\psi, \theta) : Y_1 \to K$ be two 2-shifted transverse lagrangian morphisms

$$(X_1, \eta) \xrightarrow{\phi} (K_\bullet, \omega_\bullet) \xleftarrow{\psi} (Y_1, \zeta)$$

then the fibred product

$$X_1 \times_K \overline{Y_1} := (X_1 \times_K Y_1 \rightrightarrows X_0 \times Y_0)$$

equipped with $\zeta = \mathrm{pr}_1^* \eta - \mathrm{pr}_2^* \theta$ is a quasi-symplectic groupoid.

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More generally, $(\phi, \eta) : X_1 \to K$, $(\psi, \theta) : Y_1 \to K$ be two 2-shifted lagrangian groupoids, then the homotopy fibred product

$$X_1 \widetilde{\times}_K Y_1 \cong (X_1 \times Y_1 \times K \rightrightarrows X_0 \times Y_0 \times K)$$

is quasi-symplectic

How to view multiplicative moment map reduction in this framework?

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It turns out that all the constructions of symplectic and quasi-symplectic groupoids by reduction described before are special cases of the previous result

Relation with classical moment maps

- G complete 1-connected Poisson group acting on (M, π) integrable Poisson manifold, then

$$\Sigma(M) \rightrightarrows M, \quad \mu : \Sigma(M) \to G^*, \quad G \times M \rightrightarrows M$$

are a matched pair of Lie groupoids that give us a 2-shifted lagrangian groupoid (Lu)

$$\Sigma(M) \bowtie G \rightrightarrows M, \quad (x, a) \mapsto (\mu(x), a) \in G^* \times G$$

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- (M, π) a q-Poisson G-manifold and $\mathcal{G} \rightrightarrows M$ source-simply-connected integration of $T^*_{\mathfrak{g}} M$, then $\mathcal{G}$ with $\alpha \in \Omega^2$ and $\mu : \mathcal{G} \to G$ is a *q-hamiltonian groupoid* (Li-Bland-Severa)

$$\mathcal{G} \rtimes G \rightrightarrows M, \quad (x, a) \mapsto (\mu(x)a, a) \in G \times \overline{G}, \quad \mathrm{pr}_1^* \alpha + (\mu \circ \mathrm{pr}_1, \mathrm{pr}_2)^* \omega_2$$

make it into a 2-shifted lagrangian groupoid (assuming that G-acts on $\mathcal{G}$)

Other examples of 2-shifted lagrangian groupoids can be constructed by looking at the infinitesimal level

Integrations of 2-shifted infinitesimal lagrangians

$\mathfrak{k}$ quadratic Lie algebra, $\eta_0 \in \Omega^3(X_0)$ closed

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This follows from Van Est Theorem (Arias Abad-Crainic)

Quasi-Poisson $(\mathfrak{d}, \mathfrak{g})$-manifolds

$(\mathfrak{d}, \mathfrak{g})$ Manin pair of Lie algebras,

a q-Poisson $(\mathfrak{d}, \mathfrak{g})$-manifold is an infinitesimal 2-shifted lagrangian (Bursztyn-Crainic-Severa):

a manifold M equipped with a Dirac structure

$$L \subset \mathbb{T}M \times \overline{\mathfrak{d}}, \quad L \cap (TM \oplus 0) \times \{0\} = \{0\}, \quad L \circ (TM \oplus 0) = \mathfrak{g}$$

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If we choose a lagrangian complement $\mathfrak{h} \hookrightarrow \mathfrak{d}$, the conditions above are equivalent to having (1) ρ a $\mathfrak{g}$-action on M and (2) π bivector field on M such that

$$L = \{\rho(u) + \pi\alpha \oplus \alpha, u \oplus -\rho^*(\alpha) : \alpha \in T^*M, u \in \mathfrak{g}\}$$

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(and several equations for ρ, π...)

Quasi-Poisson $(\mathfrak{d}, \mathfrak{g})$-manifolds

$(\mathfrak{d}, \mathfrak{g})$ Manin pair of Lie algebras,

a q-Poisson $(\mathfrak{d}, \mathfrak{g})$-manifold is an infinitesimal 2-shifted lagrangian (Bursztyn-Crainic-Severa):

a manifold M equipped with a Dirac structure

$$L \subset \mathbb{T}M \times \overline{\mathfrak{d}}, \quad L \cap (TM \oplus 0) \times \{0\} = \{0\}, \quad L \circ (TM \oplus 0) = \mathfrak{g}$$

If we choose a lagrangian complement $\mathfrak{h} \hookrightarrow \mathfrak{d}$, the conditions above are equivalent to having (1) ρ a $\mathfrak{g}$-action on M and (2) π bivector field on M such that

$$L = \{\rho(u) + \pi\alpha \oplus \alpha, u \oplus -\rho^*(\alpha) : \alpha \in T^*M, u \in \mathfrak{g}\}$$

(and several equations for ρ, π...)

Example:

- if $(\mathfrak{d}, \mathfrak{g}) = (\mathfrak{g} \bowtie \mathfrak{g}^*, \mathfrak{g})$ (double of a Manin triple), then L is isomorphic to a canonical Lie algebroid $T^*_\pi M \bowtie \mathfrak{g}$ associated to a Poisson $\mathfrak{g}$-action on (M, π) (Lu)

Corollary: integration of quasi-Poisson $(\mathfrak{d}, \mathfrak{g})$-manifolds

What are q-Poisson $(\mathfrak{d}, \mathfrak{g})$-manifolds globally?

L integrable integrates to 2-shifted lagrangian groupoid $(\Phi, \eta) : \mathcal{G} \to D$ such that

$$\eta_0 = 0, \quad \mathrm{Lie}(\Phi)| : \ker \eta_1^\flat :\to \mathfrak{g} \times M \text{ isomorphism}$$

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Example (Bursztyn-Iglesias-Lu): G Poisson group with Manin triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*)$. The Dirac structure $T^*_\pi G \rtimes \mathfrak{g} \hookrightarrow \mathbb{T}G \times \mathfrak{d}$ corresponding to the Poisson action of $\mathfrak{g}$ on G integrates to

$$\phi : (\mathcal{G}(D) = G \times G \times G^* \rightrightarrows G) \to D, \quad \phi(a, b, u) = a^{-1}ub$$

$$\eta_1 = \frac{1}{2}\langle \theta^l_{G,1} \wedge \theta^r_D \rangle + \frac{1}{2}\langle \theta^r_{G^*} \wedge \theta^r_{G,2} \rangle, \quad \eta_0 = 0$$

Application: integration of affine Dirac structures

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If $\mathfrak{g} \cap \mathfrak{l}$ is integrable by a closed subgroup $H \hookrightarrow G$, then $\mathcal{G}(L)$ is Morita equivalent to an integration of the Poisson homogeneous space G/H

Examples: spaces of representations

$\mathfrak{g}$ quadratic Lie algebra

- quasi- Poisson $\mathfrak{g}$-manifolds have a fusion product:

$$L_i \subset \mathbb{T}M_i \times \overline{\mathfrak{g}} \oplus \mathfrak{g}, \quad i = 1, 2 \quad \Rightarrow L_1 * L_2 \subset \mathbb{T}(M_1 \times M_2) \times \overline{\mathfrak{g}} \oplus \mathfrak{g}$$

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$$M_{\Sigma,G} = \hom(\pi_1(\Sigma), G) \quad \text{is a q-Poisson } G\text{-manifold}$$

(Alekseev-Kosmann-Schwarzbach-Meinrenken)

- Is $M_{\Sigma,G}$ integrable?

Doubles of surfaces give integrations

- $D\Sigma$ double of Σ

$$M_{D\Sigma,G} = \hom(\pi_1(D\Sigma, \{x_0, x_1\}), G) \rightrightarrows \hom(\pi_1(\Sigma), G)$$

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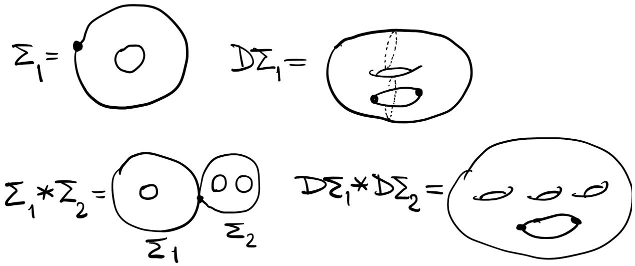
$$M_{\Sigma_1 * \Sigma_2, G} = M_{\Sigma_1, G} * M_{\Sigma_2, G},$$

- The double operation is compatible with fusion

$$M_{D(\Sigma_1 * \Sigma_2), G} = M_{D\Sigma_1, G} * M_{D\Sigma_2, G}$$



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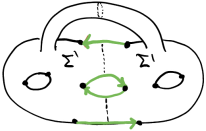
More generally, we can double a surface with multiple marked points and also produce a 2-shifted lagrangian groupoid

(Σ, V) marked surface, $V \subset \partial\Sigma$ finite, $\omega_{\Sigma,V}$ Li-Bland-Severa 2-form,
$S \subset \partial\Sigma$ a union of unmarked arcs

$\hom\Big(\pi_1\Big($ 

$, G\Big) = \hom(\pi_1(\Sigma \cup_S \Sigma, V \sqcup V), G) \xrightarrow{\ \Phi\ } G^4$

$\downarrow \downarrow$

$\hom\Big(\pi_1\Big($  $, G\Big) = \hom(\pi_1(\Sigma, V), G)$

Then $\begin{cases} \Phi(\rho) = \text{values of } \rho \text{ along } \{\leftarrow\} \\ \eta_1 = \omega_{\Sigma,V}\ , \quad \eta_0 = \text{pullbacks of Cartan 3-form along } \{\leftarrow\} \end{cases}$

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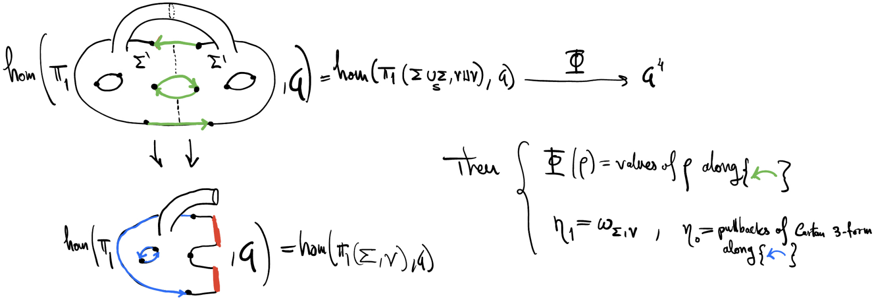

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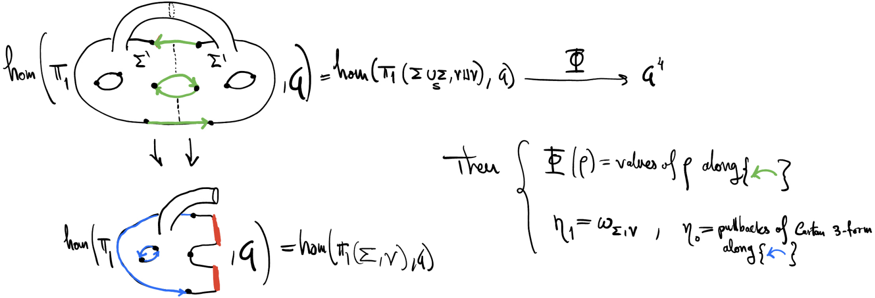

$(\Phi, \eta_1 + \eta_0)$ is a 2-shifted lagrangian structure

(this is not an integration of a q-Poisson structure in general since the 3-form component is nonzero)

Example

Take (Σ, V) the disk with two marked points, $S \subset \partial\Sigma$ an interval

- $\hom(\pi_1(\Sigma \cup_S \Sigma, V \coprod V), G) \cong (G^3 \rightrightarrows G)$ $(x, y, z) \in G^3$, $w = xyz$

$$\eta_0 = \omega_1, \quad \eta_1|_{(x,y,z)} = \frac{1}{2}\langle x^*\theta^r \wedge w^*\theta^r\rangle + \frac{1}{2}\langle y^*\theta^l \wedge z^*\theta^r\rangle$$

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- Extend the moment map Φ by using $\mathtt{s}, \mathtt{t}$ and we get still a 2-shifted lagrangian structure

$$\Psi = (\Phi, \mathtt{s}, \mathtt{t}) : (G^3 \rightrightarrows G, \eta_1 + 0) \to (G \times G \times G^2 \rightrightarrows G, \mathrm{pr}_1^*\omega - \mathrm{pr}_2^*\omega, -\delta\eta_0)$$

Decorations give us fibred products of 2-shifted lagrangians

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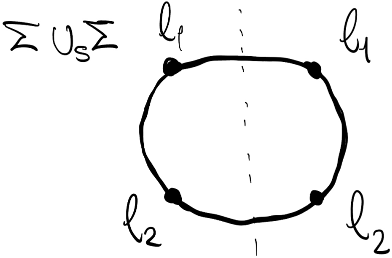
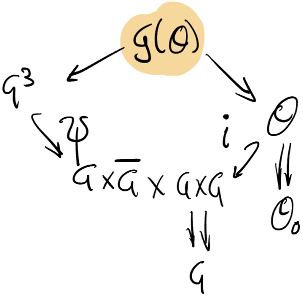
- So $\mathcal{G}(\mathcal{O}) = G^3_\Psi \times_i \overline{\mathcal{O}}$ is 1-shifted symplectic





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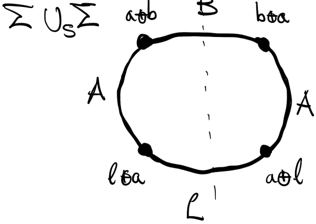

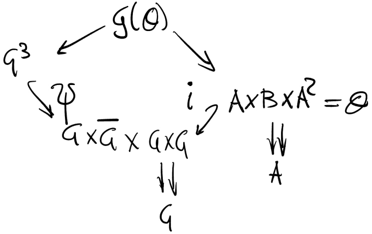

More general 2-shifted lagrangian groupoids

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and there is a 2-shifted symplectic groupoid structure on $\mathcal{G}(A)$ and a 2-shifted lagrangian structure on Ψ encoded in an exact multiplicative Courant morphism over Ψ

$$\mathbb{T}\mathcal{G}(L) \to \mathtt{t}^* E \oplus \mathtt{s}^* \overline{E}$$

Thanks!