KU LEUVEN

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Remarks on b^k-geometry

Goal: Classify Lie algebroids with a specified local model, e.g.:

Definition

A Lie algebroid $\mathcal{A}$ is of b^k-**type** if locally $\mathcal{A} = \langle x_1^k \partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n} \rangle$.

[1] builds these using defining functions, but there are more. Let $\xi \subset TM$ be a distribution, and $W \subset M$ a submanifold.

Jets of vector fields

Definition

We say that $X \in \mathfrak{X}^1(M)$ is **r-tangent to** ξ **along** W if $j^r X|_W$ is in the kernel $J^r(TM)|_W \to J^r(TM/\xi)|_W$.

E.g., $M = \mathbb{R}^n$, $W = \{x_1 = 0\}$, $\xi = \ker dx_1$, then X if $(k-1)$-tangent to ξ along W if and only if $X \in \langle x_1^k \partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n} \rangle$.

Lemma

The vector fields tangent to ξ along W are given by

$$\mathrm{Tan}^r(M, W, \xi) = I_W^r \mathfrak{X}^1(M) + \Gamma(\xi).$$

$\mathrm{Tan}^r(M, W, \xi)$ doesn't depend on the full ξ, just on $j^r \xi|_W$.

Lie algebroids

Question

When is $\mathrm{Tan}^r(M, W, \xi)$ a Lie algebroid?

- Locally free $\Leftrightarrow$ W is a hypersurface.
- Involutive $\Leftrightarrow$ ξ is involutive "up to order r".

Definition

$\sigma \in J_p^r(\mathrm{Gr}(TM, l))$ is **integrable** if for any ξ with $j_p^r \xi = \sigma$ we have that $[\Gamma(\xi), \Gamma(\xi)]$ is r-tangent to ξ at p.

The theory culminates to :

Proposition

If $\mathcal{A}$ is of b^k-type, then there exists a distribution ξ such that $\mathcal{A} = \mathrm{Tan}^{k-1}(M, Z, \xi)$ for some hypersurface Z.

Proof.

- Given $\mathcal{A}$, consider $J^{k-1}\rho(\mathcal{A})$, this will give you $j^{k-1}\xi$.
- Conversely, need local coordinates such that $\xi = \ker dx_1$. Follows from below. $\square$

Question

Suppose $j^r \xi$ is integrable, does there exist a foliation $\mathcal{F}$ with $j^r \xi = j^r \mathcal{F}$? Yes, when $W = \mathbb{R}^n$.

References

[1] The geometry of b^k-manifolds, Geoffrey Scott, 2016, JSG
[2] Poisson structures of divisor type, Ralph Klaasse, 2018.

Haefliger

We can play this game for many local models, e.g. $\left\langle x_1^{k_1} \partial_{x_1}, x_1^{k_2} \partial_{x_2}, \dots, x_1^{k_n} \partial_{x_n} \right\rangle$ will involve Lie filtrations.

Let's consider **elliptic**^{k}: $\langle r_1^k \partial_{r_1}, \partial_{\theta_1}, \partial_{x_3}, \dots, \partial_{x_n} \rangle$.

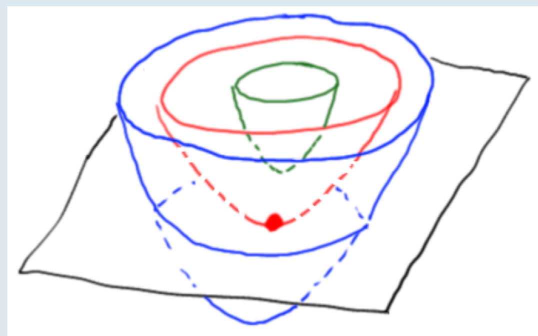

Figure: Haefliger structure inducing the singular foliation $\{x^2 + y^2 = c\}$

Construction

Let $E \to M$ be a germ of bundle, $W \subset E$ a germ of submanifold, ξ a germ of distribution around W. Consider

$$\mathrm{Tan}^{k-1}(E, W, \xi) \cap TM.$$

E.g., in the picture this will give $\langle r^k \partial_r, \partial_\theta \rangle$.

Lemma

Let $I = \langle f \rangle \subset C^\infty(M)$, and suppose that $\mathcal{A} := \{X \in \mathfrak{X}^1(M) : L_X(I) \subset I\}$ is a Lie algebroid. Then

$$\mathcal{A} = \mathrm{Tan}^1(M \times \mathbb{R}, f^{-1}(\{0\}), \ker df) \cap TM$$

Now we can repeat the story, and from every elliptic^k Lie algebroid, obtain a triple (E, W, ξ) such that $\mathcal{A} = \mathrm{Tan}^{k-1}(E, W, \xi) \cap TM$.

Poisson geometry

Proposition (Folklore)

Let f_1, f_2 be two defining functions for Z. Then $\mathrm{Tan}^r(M, Z, \ker df_1) \simeq \mathrm{Tan}^r(M, Z, \ker df_2)$.

Not the case in general, however:

Proposition [2]

If π lifts symplectically to $\mathrm{Tan}^r(M, Z, \xi)$ for some ξ, then it lifts to any $\mathrm{Tan}^r(M, Z, \xi')$.

Holonomy

The notion of holonomy survives after taking jets:

Definition

For ξ around W, let $\mathrm{hol}(\xi, W) : \Pi^1(W) \to \mathrm{Diff}(\nu(W))$ denote the holonomy, and

$$j^r \mathrm{hol}(\xi, W) : \Pi^1(W) \to J^k \mathrm{Diff}(\nu(W)),$$

the quotient to the jet groupoid.

Proposition

Let ξ, ξ' be distributions of rank l, then TFAE:

- $j^r \mathrm{hol}(\xi, W) = j^r \mathrm{hol}(\xi', W)$
- ξ is k-tangent to ξ' at W

Consequently, there is a well-defined notion of r-**holonomy** for r-jets of distributions.

Hence we have a well-defined notion of holonomy for b^k-algebroids, which is finer than that for singular foliations.