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Stability results in Poisson geometry and higher Lie theory

The setup

There is a large amount of geometric structures on a manifold M, which induce (singular) foliations on M, partitioning M into submanifolds of various dimensions called leaves.

Examples

- Poisson structures,
- Lie ($n-$)algebroids,
- Courant algebroids,
- Dirac structures.

We restrict our attention to zero-dimensional leaves, or singular points.

Definition (Stability of singular points)

Let Q be a (Poisson, Lie ($n-$)algebroid, Courant algebroid, Dirac...) structure on M inducing a partition of M as above, such that some point $p \in M$ is a zero-dimensional leaf of Q. Then $p \in M$ is stable if any structure Q' near Q of the same class as Q has a singular point $q \in M$ near p.

The problem

Question

Given a (Lie ($n-$)algebroid, Poisson, Courant...) structure Q, with a singular point $p \in M$, when is p stable?

For Poisson structures and Lie algebroid structures, a sufficient cohomological criterium was given in [1] and [3].

The approach

We first observe that the question for Poisson structures, can be phrased in an algebraic way.

Observation

Assume that $M = \mathbb{R}^n$, and $p = 0 \in \mathbb{R}^n$.

- Recall that Poisson structures are precisely $\pi \in \mathfrak{X}^2(\mathbb{R}^n)$ such that $[\pi, \pi]_{SN} = 0$, where $[-,-]_{SN}$ is the Schouten-Nijenhuis bracket.
- $\pi \in \mathfrak{X}^2(\mathbb{R}^n)$ satisfies $\pi_p = 0$ if and only if $\pi \in I_p\mathfrak{X}^2(\mathbb{R}^n)$, where I_p is the ideal of functions vanishing in p.
- For $q \in \mathbb{R}^n$, $\pi_q = 0$ if and only if $(\pi \circ T_q)_p = 0$, where $T_q : \mathbb{R}^n \to \mathbb{R}^n$ is the translation by q. Moreover, $\pi \circ T_q$ is precisely the time-1 solution to the initial value problem
$$\frac{d}{dt}\pi_t = [q, \pi_t]_{SN}, \qquad \pi_0 = \pi,$$
where $q \in \mathbb{R}^n \cong T_p\mathbb{R}^n \cong \mathfrak{X}(\mathbb{R}^n)/I_p\mathfrak{X}(\mathbb{R}^n)$ is viewed as constant vector field.

Observe that $\mathfrak{h}^\bullet := I_p\mathfrak{X}^{\bullet+1}(\mathbb{R}^n)$ is a differential graded Lie subalgebra of $(\mathfrak{g}^\bullet := \mathfrak{X}^{\bullet+1}(\mathbb{R}^n), [-,-]_{SN}, [\pi,-]_{SN})$.

This observation allows us to rephrase the question of stability as follows:

Question (Poisson stability of singular points, algebraic version)

Given $\pi' \in \mathfrak{g}^1$ near $\pi \in \mathfrak{h}^1$ with $[\pi', \pi'] = 0$, when does there exist $q \in \mathfrak{g}^0/\mathfrak{h}^0$ such that some lift $\tilde{q} \in \mathfrak{g}^0$ has the property that the solution to the initial value problem
$$\frac{d}{dt}\pi_t = [\tilde{q}, \pi_t], \qquad \pi_0 = \pi'$$
satisfies $\pi_1 \in \mathfrak{h}^1$?

Note that the latter formulation does not refer to the exact object we deal with, and only uses the algebraic structure underlying the deformation theory of Poisson structures.

[1] Marius Crainic and Rui Loja Fernandes, *Stability of symplectic leaves*, Inventiones mathematicae **180** (February 2010), no. 3, 481–533.

[2] Miquel Cueca and Rajan Amit Mehta, *Courant cohomology, Cartan calculus, connections, curvature, characteristic classes*, Commun. Math. Phys. **381** (2021), no. 3, 1091–1113 (English).

[3] Jean-Paul Dufour and Aïssa Wade, *Stability of higher order singular points of Poisson manifolds and Lie algebroids*, Ann. Inst. Fourier **56** (2006), no. 3, 545–559 (English).

Main result

We give a sufficient criterium for a postive answer to the general algebraic version of the stability question in the case that $\mathfrak{h}$ has finite codimension in $\mathfrak{g}$:

Theorem 1

Let $(\mathfrak{g}, \partial, [-,-])$ be a differential graded Lie algebra, and $\mathfrak{h} \subset \mathfrak{g}$ a differential graded Lie subalgebra of degreewise finite codimension. Let $\sigma : \mathfrak{g}^0/\mathfrak{h}^0 \to \mathfrak{g}^0$ be a splitting of the quotient map such that for any $X \in \mathfrak{g}^0/\mathfrak{h}^0$, the initial value problem
$$\frac{d}{dt}Q_t = \partial(\sigma(X)) - [\sigma(X), Q_t], \qquad Q_0 = Q' \tag{1}$$
has a unique solution up to $t = 1$, for all $Q' \in \mathfrak{g}^1$.

Let $Q \in \mathfrak{h}^1$ such that
$$\partial(Q) + \frac{1}{2}[Q, Q] = 0,$$
and assume that
$$H^1(\mathfrak{g}/\mathfrak{h}, \partial + [Q, -]) = 0.$$
Then for any $Q' \in \mathfrak{g}^1$ near Q with
$$\partial(Q') + \frac{1}{2}[Q', Q'] = 0,$$
there exists $X \in \mathfrak{g}^0/\mathfrak{h}^0$ such that the time-1 solution of (1) lies in $\mathfrak{h}^1$.

Applications

Examples

Unless otherwise stated, $\partial = 0$, and all vector bundles are over $\mathbb{R}^n$. Let $p \in \mathbb{R}^n$ be the origin.

- **Singular points of order $k \geq 1$ of a Poisson structure π:** Take
$$\mathfrak{g}^\bullet := \mathfrak{X}^{\bullet+1}(\mathbb{R}^n),\ \mathfrak{h}^\bullet := I_p^{1+\bullet(k-1)}\mathfrak{X}^{\bullet+1}(\mathbb{R}^n) \text{ and } Q = \pi \in \mathfrak{h}^1,$$
with the Schouten-Nijenhuis bracket, where I_p is the vanishing ideal of the origin. For $k = 1$, Theorem 1 is then equivalent to [1, Thm. 1.1] for zero-dimensional leaves, and for general k, it is equivalent to [3, Thm. 1.2].
- **Singular points of order (k, l) of a Lie n-algebroid structure Q_0 on $E = \bigoplus_{i=1}^n E_i[1]$:** Take
$$\mathfrak{g}^\bullet := \mathrm{Der}_{\mathbb{R}}(S(E^*[-1])) \text{ and } Q = Q_0 \in \mathfrak{h}^1,$$
with the commutator bracket, and let $\mathfrak{h}^\bullet$ be the subspace of derivations with certain vanishing behaviour in the origin. For $n = 1$, l is irrelevant, and we deal with singular points of order k of Lie algebroids as in [3], and Theorem 1 is equivalent to [3, Thm. 1.3]. When additionally, $k = 1$, Theorem 1 is equivalent to [1, Thm. 1.8] for singular points.
- **Singular points of a Courant algebroid structure Θ on $(E, \langle -,- \rangle)$:** Take
$$\mathfrak{g}^\bullet := C^{\bullet+2}(E) \text{ and } Q = \Theta$$
as in [2]. For $\mathfrak{h}^\bullet$ we restrict to subspaces corresponding to vanishing symbol at p. The cohomology is then isomorphic to a subspace of $H^1_{CE}(\mathfrak{g}_p, \mathbb{R}^n)$ where $\mathfrak{g}_p$ is the quadratic Lie algebra obtained by restricting the Courant algebroid to p acting on $T_p\mathbb{R}^n$ by the Bott representation, where the subspace is defined by the pairing.
- **Singular points of a Dirac structure $L \subset L \oplus L^*$ with L^* Dirac:** Assume that the origin $p \in \mathbb{R}^n$ is a singular point of L. Take
$$\mathfrak{g}^\bullet := \Gamma(\wedge^{\bullet+1}L^*),$$
with the bracket induced by the Lie algebroid structure of L^*, and $\partial = d_L$.Then take
$$\mathfrak{h}^\bullet := \{X \in \Gamma(\wedge^{\bullet+1}L^*) \mid X_p \in \wedge^{\bullet+1}\ker((\rho_{L^*})_p)\} \text{ and } Q := 0.$$
The cohomological assumption is given by
$$\frac{\ker(d_L : \wedge^2 L_p^*/\wedge^2 \ker((\rho_{L^*})_p) \to \wedge^3 L_p^*/\wedge^3 (\ker(\rho_{L^*})_p))}{\mathrm{im}(d_L : L_p^*/\ker((\rho_{L^*})_p) \cong T_pS \to \wedge^2 L_p^*/\wedge^2 \ker((\rho_{L^*})_p))} = 0.$$
This cohomological condition even implies the existence of a singular point on the L^*-leaf through p.

For the future...

Try to get rid of finite codimension assumption, for stability of higher-dimensional leaves, (transitive) Lie subalgebroids, representations (up to homotopy) of Lie algebroids...