Classical gauge theories on E-manifolds

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Abstract

We extend the formalism of vector bundles, principal bundles and principal connections to E-manifolds (as introduced in [6]), which can be used to describe singularities in the configuration space of a classical particle. Manifolds with boundary or corners are configuration spaces naturally described in terms of E-manifolds extending that of b-manifolds [2]. Following [8], we show the existence of a universal model for the phase space of a particle interacting with a gauge field; in this new setting, Wong's equations become hamiltonian. Also following [5], we see that the universal E-symplectic spaces of Weinstein are symplectic leaves of a bigger universal Poisson space.

The geometry of E-manifolds

Definition 1 (E-manifold) An *E-manifold* is a pair (M, E), where M is a smooth manifold and $E \subseteq \mathrm{Vec}(M)$ is an involutive and locally free $\mathcal{C}^\infty(M)$-submodule. We call any such $E \subseteq \mathrm{Vec}(M)$ an *E-structure*, and write ${}^E\mathrm{Vec}(M) = E$. ◂

Lemma 2 (E-tangent bundle characterization) Let (M, E) be an E-manifold. There exists a vector bundle ${}^E\mathrm{T}M$, called the *E-tangent bundle*, such that local sections of ${}^E\mathrm{T}M$ are in one-to-one correspondence with local sections in E. ◂

Definition 3 (E-maps) If (M, E_M) and (N, E_N) are E-manifolds, an *E-map* is a Lie algebroid morphism $F\colon {}^E\mathrm{T}M \longrightarrow {}^E\mathrm{T}N$. ◂

Example 4 (b-Manifolds) A *b-manifold* is a pair (M, Z), where M is a smooth manifold and $Z \subset M$ is an embedded hypersurface. The submodule of tangent vector fields to the submanifold Z is an E-structure. In a coordinate chart (U, φ) with coordinates $q_1, \ldots, q_n$ adapted to Z,

$$q_1\frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \ldots, \frac{\partial}{\partial q_n}$$

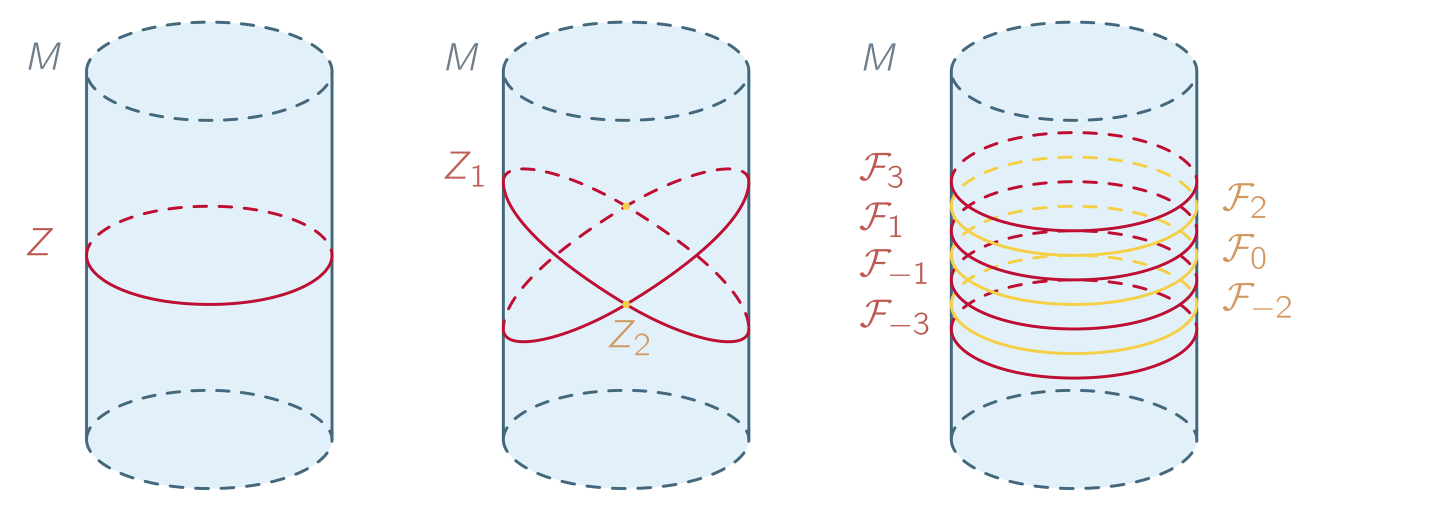
are generators of b-fields. ◂

Example 5 (c-Manifolds) A *c-manifold* is a pair (M, Z), where M is a smooth manifold M and $i\colon Z \longrightarrow M$ is an immersed hypersurface with self-transverse intersections. The tangent fields to $i(Z)$ are an E-structure. There exist coordinates $\boldsymbol{q}$ such that $i(Z) = \cup_{i \leq k}\{q_i = 0\}$. We have generators

$$q_1\frac{\partial}{\partial q_1}, \ldots, q_k\frac{\partial}{\partial q_k}, \frac{\partial}{\partial q_{k+1}}, \ldots, \frac{\partial}{\partial q_n}.$$ ◂

Example 6 (Regular foliations) Consider a smooth manifold M and a regular, smooth and involutive distribution $\mathcal{D}$ of rank k. By Frobenius' theorem, there exists a foliation $\mathcal{F}$ such that any element of $\mathcal{D}$ is tangent to a leaf of $\mathcal{F}$. The distribution $\mathcal{D}$ defines an E-structure by the involutivity condition. A choice of coordinates $\boldsymbol{q}$ adapted to the foliation $\mathcal{F}$ gives a local basis

$$\frac{\partial}{\partial q_1}, \ldots, \frac{\partial}{\partial q_k}.$$ ◂



Preliminaries on gauge theories

Definition 7 (Prolongation, pullback structure) Let (M, E_M) be an E-manifold and consider a fibre bundle $\tau\colon B \longrightarrow M$. The *prolongation* of E by B is the pullback submodule $E_B = \langle \tau_*^{-1}(E_M)\rangle$. ◂

Definition 8 (Liouville one-form) Let (M, E_M) be an E-manifold and consider the prolongation $({}^E\mathrm{T}^*M, \tau_*^{-1}(E_M))$. The *Liouville form* $\lambda \in {}^E\Omega({}^E\mathrm{T}^*M)$ is defined by its action on $X \in {}^E\mathrm{Vec}({}^E\mathrm{T}^*M)$ as

$$\langle \lambda, X\rangle = \left\langle \tau_{{}^E\mathrm{T}{}^E\mathrm{T}^*M}(X), (\tau_{{}^E\mathrm{T}^*M})_*(X)\right\rangle.$$ ◂

Definition 9 (Canonical symplectic form) In the previous notation, the *canonical symplectic form* is defined as $\omega = \mathrm{d}\lambda$. Moreover, in natural coordinates

$$\omega = \sum_{i=1}^{p} V_i^* \wedge E_i^* - \frac{1}{2}\sum_{i,j,k=1}^{p} r_i C^i_{jk} E_j^* \wedge E_k^*.$$ ◂

Definition 10 (E-principal connection, E-gauge field) Consider an E-manifold (M, E_M) and a principal G-bundle $\pi\colon P \longrightarrow M$ with pullback structure ${}^E\mathrm{T}P$. An *E-principal connection* is a G-invariant splitting of the short exact sequence

$$0 \longrightarrow P \times \mathfrak{g} \xrightarrow{\iota} {}^E\mathrm{T}P \xrightarrow{\pi_*} \pi^*{}^E\mathrm{T}M \longrightarrow 0,$$

called the *E-Atiyah sequence*. ◂

The formalisms of Weinstein and Montgomery

Theorem 11 (Weinstein's symplectic formulation [8]) *Consider a principal G-bundle $\pi\colon P \longrightarrow M$ over an E-manifold M and a Hamiltonian G-space Q.*

1. *The product space ${}^E\mathrm{T}^*P \times Q$ is Hamiltonian with moment map $\mu_P + \mu_Q$.*
2. *The hypotheses of the reduction theorem are satisfied and, consequently, the space $({}^E\mathrm{T}^*P \times Q)_0$ is an E-symplectic manifold.*
3. *The horizontal lift $h^\dagger$ is well defined in classes of equivalence and defines a map $\alpha\colon ({}^E\mathrm{T}^*P \times Q)_0 \longrightarrow {}^E\mathrm{T}^*M$.* ◂

Given a function $H \in \mathcal{C}^\infty({}^E\mathrm{T}^*M)$, the pullback $\alpha^*H \in \mathcal{C}^\infty(({}^E\mathrm{T}^*M \times Q)_0)$ is a hamiltonian function for *Wong's equations of motion*.

Theorem 12 (Weinstein's isomorphism with Sternberg's space [7, 8]) *Let $\pi\colon P \longrightarrow M$ be an E-principal G-bundle and consider a G-Hamiltonian space Q. There exists a diffeomorphism*

$$\mu^{-1}(0) \simeq P^\sharp \times Q.$$

The map descends to a symplectomorphism $({}^E\mathrm{T}^*P \times Q)_0 \simeq P^\sharp \times_G Q$. ◂

Theorem 13 (Montgomery's isomorphism [5]) *Consider an E-principal G-bundle $\pi\colon P \longrightarrow M$. Any E-principal connection gives rise to the commutative diagram (1). Moreover, the map $[\Psi]$, called the* minimal coupling, *is a Poisson isomorphism. The Weinstein space $({}^E\mathrm{T}^*P \times \mathcal{O}(p))_0$ is a symplectic leaf of the Poisson space ${}^E\mathrm{T}^*P/G$.*

$$\begin{tikzcd}
{}^E\mathrm{T}^*P \arrow[r, "\pi_{{}^E\mathrm{T}^*P/G}"] & {}^E\mathrm{T}^*P/G \arrow[dr, "{[h^\dagger]}"] & \\
 & & {}^E\mathrm{T}^*M \\
\mathfrak{g}^* \times P^\sharp \arrow[uu, "\Psi"] \arrow[r, "\pi_{\mathfrak{g}^* \times_G P^\sharp}"'] & \mathfrak{g}^* \times_G P^\sharp \arrow[uu, "{[\Psi]}"] \arrow[ur, "{[\pi_2]}"'] &
\end{tikzcd} \tag{1}$$ ◂

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