Courant-Nijenhuis algebroids

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Introduction

Inspired by Poisson-Nijenhuis structures [4], we define Courant-Nijenhuis algebroids using the formalism of generalized derivations on vector bundles. From this approach, we introduce a notion of Dirac-Nijenhuis structures which recovers the definition of Poisson-Nijenhuis structures when the Courant algebroid is the standard $TM \oplus T^*M$. We also revisit the theory of Manin triples for Lie algebroids to establish a relation between Lie-Nijenhuis bialgebroids, Courant-Nijenhuis algebroids and Dirac-Nijenhuis structures.

Motivation: Poisson-Nijenhuis structures

The Poisson-Nijenhuis (PN) structures was introduced in the context of integrable systems by Magri and Morosi [4]. They showed that it is possible to produce an integration algorithm for bi-hamiltonian systems described by such structures. Several aspects of PN structures were studied, but more than the definition, the work of Stienon and Xu is of particular for our purpose. They show that PN structures are the real part of holomorphic Poisson structures.

A **PN structure** on a manifold M is a pair (π, r), where $\pi \in \mathfrak{X}^2(M)$ is a Poisson structure and $r : TM \to TM$ is a Nijenhuis tensor, and they are **compatible:**

1. $\pi^\sharp \circ r^* = r \circ \pi^\sharp$,
2. The tensor field $R(\pi, r)$ vanishes, where

$$R(\pi,r)(X,\alpha) = (\mathcal{L}_{\pi^\sharp(\alpha)} r)(X) - \pi^\sharp(\mathcal{L}_X(r^*\alpha) - \mathcal{L}_{r(X)}(\alpha)),\ X \in \mathfrak{X}(M), \alpha \in \Omega^1(M).$$

A natural way to extend this structure for the context of Dirac structures and Courant algebroids is by extending the compatibility conditions.

Main tool: 1-derivation on vector bundles

A **1-derivation** on a vector bundle $E \to M$ is a triple (D, l, r), where $r : TM \to TM$ and $l : E \to E$ are vector bundle maps covering the identity and $D : \Gamma(E) \to \Gamma(T^*M \otimes E)$ is a $\mathbb{R}$-linear map satisfying the Leibniz-type equation

$$D_X(f\sigma) = f D_X(\sigma) + (\mathcal{L}_X f)\, l(\sigma) - (\mathcal{L}_{r(X)} f)\, \sigma.$$

Examples:

- ▷ A **connection** ∇ on E defines a 1-derivation by $r = 0$, $l = \mathrm{id}_E$ and $D = \nabla$.
- ▷ A **holomorphic structure** on E is a triple (∇, l, r), where ∇ is a flat $T^{0,1}$-connection on E, $l : E \to E$ is the fiberwise multiplication by i and $r : TM \to TM$ is a complex structure on M. The triple (∇, l, r) is equivalent to the 1-derivation (D^{hol}, l, r), where

$$D^{hol}_X(\sigma) = l(\nabla_{X + \mathrm{i}r(X)}\sigma).$$

- ▷ Given a (1,1)-tensor $r : TM \to TM$ on M, we can define a corresponding 1-derivations on TM and T^*M by (D^r, r, r) and $(D^{r,*}, r^*, r)$, respectively; where

$$D^r_X(Y) = (\mathcal{L}_Y r)(X) = [Y, r(X)] - r([Y, X])$$
$$D^{r,*}_X(\alpha) = \mathcal{L}_X(r^*\alpha) - \mathcal{L}_{r(X)}\alpha.$$

These 1-derivations on TM and T^*M are in 1:1 correspondence with the **tangent lift** and the **cotangent lift** of $r : TM \to TM$, respectively. In fact, there is a 1:1 correspondence between 1-derivations and linear $(1,1)$−tensors, see [1, 3].

PN compatibility conditions under 1-derivation viewpoint

Using the generalized 1-derivations, the compatibility conditions for PN structures become

$$r \circ \pi^\sharp = \pi^\sharp \circ r^*$$
$$D^r_X \circ \pi^\sharp = \pi^\sharp \circ D^{r,*}_X$$

which we can be understand as the commutativity of the following diagrams

$$\begin{array}{ccc} T^*M & \xrightarrow{\pi^\sharp} & TM \\ \downarrow{\scriptstyle r^*} & & \downarrow{\scriptstyle r} \\ T^*M & \xrightarrow[\pi^\sharp]{} & TM \end{array} \qquad \begin{array}{ccc} \Omega^1(M) & \xrightarrow{\pi^\sharp} & \mathfrak{X}(M) \\ \downarrow{\scriptstyle D^{r,*}_X} & & \downarrow{\scriptstyle D^r_X} \\ \Omega^1(M) & \xrightarrow[\pi^\sharp]{} & \mathfrak{X}(M) \end{array}$$

Then, we can say that behind the compatibility equations, there is a compatibility between π and 1-derivations (D^r, r, r) and $(D^{r,*}, r^*, r)$.

Main definition

Let $(E, \Pi, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!])$ be an arbitrary Courant algebroid. A **Courant 1-derivation** is a 1-derivation (D, l, r) on E such that $D^* = D$ (under the isomorphism $E \cong E^*$ coming from the pairing) and the following set of compatibility equations are satisfied

$$\begin{aligned} \Pi \circ l &= r \circ \Pi, \\ \Pi(D_X(\sigma)) &= D^r_X(\Pi(\sigma)), \\ l([\![\sigma_1,\sigma_2]\!]) &= [\![\sigma_1, l(\sigma_2)]\!] - D_{\Pi(\sigma_2)}(\sigma_1) - \Pi^*(C(\sigma_1,\sigma_2)), \\ D_X([\![\sigma_1,\sigma_2]\!]) &= [\![\sigma_1, D_X(\sigma_2)]\!] - [\![\sigma_2, D_X(\sigma_1)]\!] + D_{[\Pi(\sigma_2),X]}(\sigma_1) \\ &\quad - D_{[\Pi(\sigma_1),X]}(\sigma_2) - \Pi^*(i_X\, dC(\sigma_1,\sigma_2)), \end{aligned}$$

for all $\sigma_1, \sigma_2 \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$, where $C(\sigma_1,\sigma_2) \in \Omega^1(M)$ is defined by $C = \langle D_{(\cdot)}(\sigma_1), \sigma_2\rangle \in \Omega^1(M)$. These equations are called **Courant compatibility equations**, and describe the compatibility between the Courant algebroid structure and a 1-derivation on E.

A **Courant-Nijenhuis (CN) structure** on E is a Courant 1-derivation (D, l, r) satisfying the Nijenhuis equations:

$$N_r = 0,\ \ [D, l] = 0 \ \text{ and } \ D^2 = 0.$$

Examples:

- ▷ **Standard Courant algebroid.** The 1-derivation $\mathcal{D} = (\mathbb{D}^r, (r, r^*), r)$ on the standard Courant algebroid $\mathbb{T}M$, where

$$\mathbb{D}^r = (D^r, D^{r,*})$$

is a Courant 1-derivation. If $N_r = 0$, then $\mathcal{D}$ is a Courant-Nijenhuis structure on $\mathbb{T}M$.

- ▷ **Holomorphic Courant algebroids** Let $E \to M$ a (real) vector bundle with holomorphic structure (∇, l, r). We have that E is a holomorphic Courant algebroid if and only if (D^{hol}, l, r) is a Courant-Nijenhuis structure on E.

Dirac-Nijenhuis structures on $\mathbb{T}M$

Let $L \subset \mathbb{T}M$ be a Dirac structure and $r : TM \to TM$ be a Nijenhuis tensor. The pair (L, r) is called a **Dirac-Nijenhuis (DN) structure** on $\mathbb{T}M$ if

1. $(r, r^*)(L) \subseteq L$
2. $\mathbb{D}_X(\Gamma(L)) \subset \Gamma(L),\ \forall X \in \mathfrak{X}(M)$.

Examples:

- ▷ **PN structures.** Consider $\pi \in \mathfrak{X}^2(M)$ a Poisson structure and $L_\pi = \mathrm{graph}(\pi^\sharp)$. The pair (L_π, r) is a Dirac-Nijenhuis structure if and only if (π, r) is a PN-structure.
- ▷ **ΩN structures.** Consider $\omega \in \Omega^2(M)$ a closed 2−form and $L_\omega = \mathrm{graph}(\omega^\flat)$. A presymplectic-Nijenhuis (ΩN) structure on M is a compatible pair (ω, r), where ω is presymplectic and r is Nijenhuis and they satisfies

$$\omega^\flat \circ r = r^*\omega^\flat \ \text{ and } \ d\omega_r = (d\omega)_r.$$

The pair (L_ω, r) is a Dirac-Nijenhuis structure if and only if (ω, r) is a ΩN-structure.

- ▷ **Holomorphic Dirac structures.** Let (M, r) be a complex manifold, and consider the isomorphism $\mathbb{T}M \xrightarrow{\Phi} T_{10} \oplus (T_{10})^*$, where $\Phi(X,\alpha) = \frac{1}{2}(x - ir(x), \alpha - ir^*(\alpha))$. The pair (L, r) is a Dirac-Nijenhuis structure on $\mathbb{T}M$ if and only if $\Phi(L)$ is a holomorphic Dirac structure on $T_{10} \oplus (T_{10})^*$.

Those examples extend the definitions of PN and ΩN structures introduced in [4]. Moreover, the last one extends the relation between PN and holomorphic Poisson structures, presented in [6].

Remark: There are considerable results about Dirac-Nijenhuis structures. For example, the leaves are presymplectic-Nijenhuis, the quotient are Poisson-Nijenhuis, we have hierarchies of DN structures. Moreover, there are also integration results to a presymplectic-Nijenhuis groupoid and, as a corollary, integration of holomorphic Dirac structures (see [2]).

Manin-Nijenhuis triples

There is a relation between the Lie bialgebroid (A, A^*) and the Courant algebroid $A \oplus A^*$ (see [5]). Moreover, Drummond recently introduce a notion of Lie-Nijenhuis bialgebroids as the infinitesimal counterpart of a PN groupoid [3]. With those information in mind, We proved the following

Lie-Nijenhuis bialgebroid ⇔ Courant-Nijenhuis algebroid + Pair of transversal DN structures.

References

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