# Palatini formalism in the framework of Courant algebroids

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## 1. Abstract

Palatini approach to the Einstein-Hilbert action is a way how to justify the use of the Levi-Civita affine connection for general relativity. It has turned out that kind of similar thing can be done also for the Levi-Civita Courant algebroid connections. In the particular case of exact Courant algebroids, this procedure leads precisely to the Courant algebroid formulation of the supergravity (SUGRA) equations.

## 2. Courant algebroids

### Definition: Courant algebroid (CA)

*Courant algebroid is a 4-tuple $(E \xrightarrow{\pi} M, \rho, [\cdot,\cdot]_E, \langle\cdot,\cdot\rangle_E)$ consisting of*

- *a vector bundle $E \xrightarrow{\pi} M$,*
- *a vector bundle morphism $\rho : E \to TM$,*
- *an $\mathbb{R}$-bilinear map $[\cdot,\cdot]_E : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$,*
- *a fiber-wise metric $\langle\cdot,\cdot\rangle_E$ on $E$,*

*such that the following axioms are satisfied:*

- $[\psi_1, f\psi_2]_E = (\rho(\psi_1)f)\psi_2 + f[\psi_1,\psi_2]_E$,
- $[\psi_1,[\psi_2,\psi_3]_E]_E = [[\psi_1,\psi_2]_E,\psi_3]_E + [\psi_2,[\psi_1,\psi_3]_E]_E$,
- $\rho(\psi_1)\langle\psi_2,\psi_3\rangle_E = \langle[\psi_1,\psi_2]_E,\psi_3\rangle_E + \langle\psi_2,[\psi_1,\psi_3]_E\rangle_E$,
- $\langle[\psi_1,\psi_1]_E,\psi_2\rangle_E = \frac{1}{2}\rho(\psi_2)\langle\psi_1,\psi_1\rangle_E$,

*for all $\psi_1$, $\psi_2$, $\psi_3 \in \Gamma(E)$ and for all $f \in C^\infty(M)$.*

- Courant algebroids over the point are precisely the quadratic Lie algebras.

## 3. Generalized metrics

### Definition: Generalized metric

*Generalized metric is a maximal positive definite subbundle $V_+ \subseteq E$ with respect to $\langle\cdot,\cdot\rangle_E$.*

- Every generalized metric $V_+ \subseteq E$ induces the vector bundle decomposition $E = V_+ \oplus V_-$, where $V_- := V_+^\perp$ with respect to $\langle\cdot,\cdot\rangle_E$.
- Every generalized metric $V_+ \subseteq E$ can be unambiguously associated with a positive definite fiber-wise metric $G$ on $E$ satisfying $\flat_G \sharp_E \flat_G = \flat_E$ and vice versa.

## 4. Courant algebroid connections

### Definition: Courant algebroid connection (CA connection)

*Courant algebroid connection is an $\mathbb{R}$-bilinear map $\nabla : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ satisfying*

- $\nabla_{f\psi_1}\psi_2 = f\nabla_{\psi_1}\psi_2$,
- $\nabla_{\psi_1}(f\psi_2) = (\rho(\psi_1)f)\psi_2 + f\nabla_{\psi_1}\psi_2$,
- $\rho(\psi_1)\langle\psi_2,\psi_3\rangle_E = \langle\nabla_{\psi_1}\psi_2,\psi_3\rangle_E + \langle\psi_2,\nabla_{\psi_1}\psi_3\rangle_E$.

*for all $\psi_1$, $\psi_2$, $\psi_3 \in \Gamma(E)$ and for all $f \in C^\infty(M)$. We write $\nabla_{\psi_1}\psi_2 := \nabla(\psi_1,\psi_2)$.*

- **Divergence**

$$\mathrm{div}_\nabla \psi := \mathrm{Tr}_E \langle\star, \nabla(\star,\psi)\rangle_E. \tag{1}$$

- **Torsion tensor**, (Gualtieri, 2007)

$$T_\nabla(\psi_1,\psi_2,\psi_3) := \langle\nabla_{\psi_1}\psi_2 - \nabla_{\psi_2}\psi_1 - [\psi_1,\psi_2]_E, \psi_3\rangle_E + \langle\nabla_{\psi_3}\psi_1,\psi_2\rangle_E. \tag{2}$$

- **Riemann tensor**, (Hohm & Zweibach, 2013)

$$\begin{aligned} R_\nabla(\psi_1,\psi_2,\psi_3,\psi_4) := \frac{1}{2}\Big(&\langle\nabla_{\psi_3}\nabla_{\psi_4}\psi_2 - \nabla_{\psi_4}\nabla_{\psi_3}\psi_2 - \nabla_{[\psi_3,\psi_4]_E}\psi_2,\psi_1\rangle_E \\ &+ \langle\nabla_{\psi_2}\nabla_{\psi_1}\psi_3 - \nabla_{\psi_1}\nabla_{\psi_2}\psi_3 - \nabla_{[\psi_2,\psi_1]_E}\psi_3,\psi_4\rangle_E \\ &+ \mathrm{Tr}_E\left(\langle\nabla(\star,\psi_3),\psi_4\rangle_E\langle\nabla(\star,\psi_2),\psi_1\rangle_E\right)\Big). \end{aligned} \tag{3}$$

- **Ricci tensor**

$$\mathrm{Ric}_\nabla(\psi_1,\psi_2) := \mathrm{Tr}_E R_\nabla(\star,\psi_1,\star,\psi_2). \tag{4}$$

## 5. Interplay of generalized metrics and CA connections

For a CA equipped with a generalized metric and a CA connection, the following can be defined.

- **Ricci scalar** corresponding to $G$

$$\mathcal{R}_{\nabla,G} := \mathrm{Tr}_G \mathrm{Ric}_\nabla. \tag{5}$$

- $\nabla$ is **Ricci compatible** with $V_+$ if and only if $\mathrm{Ric}_\nabla|_{V_+\times V_-} = 0$.
- $\nabla$ is **Levi-Civita CA connection** with respect to $V_+$ if and only if $T_\nabla = 0$, and moreover, for all $\psi \in \Gamma(E)$ there holds $\nabla_\psi(\Gamma(V_+)) \subseteq \Gamma(V_+)$.

### Proposition: Existence and ambiguity of Levi-Civita CA connection

*Let $(E \xrightarrow{\pi} M, \rho, [\cdot,\cdot]_E, \langle\cdot,\cdot\rangle_E)$ be a CA, and let $V_+ \subseteq E$ be a generalized metric. Levi-Civita CA connection with respect to $V_+$ exists. Moreover, except for the case $p, q \in \{0,1\}$, where $(p,q)$ denotes the signature of $\langle\cdot,\cdot\rangle_E$, there are infinitely many of L-C CA connections w.r.t. $V_+$.*

## 6. Standard Palatini formalism, (Einstein, 1925)

Standard Palatini formalism is an alternative variational formulation of general relativity. The difference from the standard one is that a metric and an affine connection are treated as two mutually independent dynamical fields.

### Theorem: Palatini approach to the Einstein-Hilbert action

*Let $M$ be an oriented connected manifold with $\dim(M) \neq 2$. Then an affine connection $\nabla$ and a (semi-)Riemannian metric $g$ on $M$ extremalize the Einstein-Hilbert action*

$$S[\nabla, g] := \int_M \mathcal{R}^{\nabla,g}\,\mathrm{Vol}_g \tag{6}$$

*if and only if*

1. $\nabla = \nabla^{LC,g} + \omega \otimes \mathrm{Id}_{\mathfrak{X}(M)}$ *for some* $\omega \in \Omega^1(M)$,
2. *Einstein vacuum field equation without cosmological constant holds, i.e.*

$$[\mathrm{Ric}^\nabla]_S - \frac{1}{2}\mathcal{R}^{\nabla,g} g = 0. \tag{7}$$

It can be shown that neither equation (7) nor the geodesic equation depends on the particular choice of $\omega$. Therefore, without changing any observable physics, we are able to impose $\omega = 0$, i.e. to choose $\nabla = \nabla^{LC,g}$.

## 7. "Generalized" Palatini formalism, (Moučka & Vysoký, 2022)

"Generalized" Palatini formalism is an observation justifying the concept of Levi-Civita CA connections and the form of the torsion tensor.

### Lemma: CA connection compatible with volume form

*Let $(E \xrightarrow{\pi} M, \rho, [\cdot,\cdot]_E, \langle\cdot,\cdot\rangle_E)$ be a Courant algebroid over an oriented connected manifold, and let $\mathrm{Vol}$ be a volume form on $M$. Then the following statements hold:*

1. *For an arbitrary Riemannian metric $g$ on $M$ there is a unique smooth function $\phi \in C^\infty(M)$ such that either $\mathrm{Vol} = \mathrm{e}^{-2\phi}\,\mathrm{Vol}_g$ or $\mathrm{Vol} = -\,\mathrm{e}^{-2\phi}\,\mathrm{Vol}_g$.*
2. *If the signature $(p,q)$ of $\langle\cdot,\cdot\rangle_E$ satisfies $p \neq 1$ and $q \neq 1$, there is a L-C CA connection $\nabla$ such that*

$$\mathrm{div}_\nabla = (\mathrm{div}^g - 2\,\mathrm{d}\,\phi) \circ \rho, \tag{8}$$

*where $\mathrm{div}^g$ is the covariant divergence corresponding to the L-C affine connection $\nabla^{LC,g}$.*

*Moreover, the expression $\mathrm{div}^g - 2\,\mathrm{d}\,\phi$ does not depend on the particular choice of an auxiliary Riemannian metric $g$. Therefore, it is fully determined exclusively by $\mathrm{Vol}$, we denote it by $\mathrm{div}_{\mathrm{Vol}}$.*

### Theorem: Palatini approach to the Courant-Einstein-Hilbert action

*Consider a CA $(E \xrightarrow{\pi} M, \rho, [\cdot,\cdot]_E, \langle\cdot,\cdot\rangle_E)$ over an oriented connected manifold such that the signature $(p,q)$ of $\langle\cdot,\cdot\rangle_E$ satisfies $p \neq 1$ and $q \neq 1$. Then a volume form $\mathrm{Vol}$ on $M$, a generalized metric $G$ on $E$ and a CA connection $\nabla$ on $E$ extremalize the Courant-Einstein-Hilbert action*

$$S[\mathrm{Vol}, G, \nabla] := \int_M \mathcal{R}_{\nabla,G}\,\mathrm{Vol} \tag{9}$$

*if and only if*

1. $\mathcal{R}_{\nabla,G} = 0$,
2. $\nabla$ *is Ricci compatible with* $G$,
3. $\nabla$ *is Levi-Civita with respect to $G$, and moreover,* $\mathrm{div}_\nabla = \mathrm{div}_{\mathrm{Vol}} \circ \rho$.

- All dynamical fields ($\mathrm{Vol}$, $G$, $\nabla$) of the above action are a priori mutually independent.
- CA connection is not determined uniquely by the EOM. However, the EOM are invariant under its particular choice, and moreover, they are also invariant under so called "metric and connection preserving CA isomorphisms".

How to vary the dynamical fields?

- **Volume form**, for an arbitrary compactly supported smooth function $f$ impose

$$\mathrm{Vol}' := (1 + \epsilon f)\,\mathrm{Vol}\,. \tag{10}$$

- **Generalized metric**, for an arbitrary compactly supported vector bundle morphism $\Psi : V_+ \to V_-$ impose

$$V'_+ := \bigsqcup_{p\in M}\{v + \epsilon\Psi v \mid v \in V_{+p}\} \subseteq E. \tag{11}$$

- **CA connection**, for an arbitrary $N \in \Omega^1(E) \otimes \Omega^2(E)$ satisfying $N|_{\partial M} = 0$ impose

$$\nabla'_{\psi_1}\psi_2 := \nabla_{\psi_1}\psi_2 + \epsilon\sharp_E N(\psi_1,\psi_2,.). \tag{12}$$

For a practical calculation, it turned out to be convenient to make one step further and parametrize $\nabla$ as

$$\nabla_{\psi_1}\psi_2 = \nabla^0_{\psi_1}\psi_2 + \sharp_E L(\psi_1,\psi_2,.), \tag{13}$$

where $\nabla^0$ is a Levi-Civita CA connection satisfying $\mathrm{div}_\nabla = \mathrm{div}_{\mathrm{Vol}} \circ \rho$, and $L \in \Omega^1(E) \otimes \Omega^2(E)$.

## 8. Examples

The EOM derived from "generalized" Palatini formalism are related to various physical field theories.

- For exact CAs the EOM precisely coincide with the **supergravity equations**. The metric and B-field are encoded in generalized metric and the dilation is encoded in volume form.
- For a special class of transitive CAs, the formalism leads to the **heterotic supergravity equations**.
- We are currently looking for a way how to fit the formalism to the **generalized supergravity equations** (Tseytlin & Wulff, 2016).

## 9. Selected references

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