Quasifolds as groupoids and as diffeological spaces

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Main Result

- An effective quasifold groupoid is a Lie groupoid that is locally isomorphic to action groupoids of effective countable affine actions on $\mathbb{R}^n$.
- A diffeological quasifold is a diffeological space that is locally diffeomorphic to quotients of $\mathbb{R}^n$ by countable affine actions.

Theorem. Two effective quasifold groupoids are Morita equivalent if and only if their quotient spaces are diffeomorphic.

Joint work with Yael Karshon, based on earlier work by Masrour Zoghi.

Motivation

Toric symplectic spaces

Polytope	Symplectic toric
Rational and Delzant	Manifold
Rational and simple	Orbifold
Irrational and simple	Quasifold.

Prato in 1999 introduced quasifolds to complete the table above.

Irrational tori

$$T_\alpha := \mathbb{R}/(\mathbb{Z}+\alpha\mathbb{Z}) \text{ is } \begin{cases} \text{a torus if } \alpha \text{ is rational} \\ \text{an irrational torus if } \alpha \text{ is irrational.} \end{cases}$$

Donato and Iglesias-Zemmour, 1985:

$$T_\alpha \text{ is diffeomorphic to } T_\beta \iff \alpha \equiv \beta \mod \mathrm{GL}(2;\mathbb{Z}).$$

- Groupoid: $(\mathbb{Z}+\alpha\mathbb{Z}) \ltimes \mathbb{R}$ (action groupoid).
- Diffeological space: $\mathbb{R}/(\mathbb{Z}+\alpha\mathbb{Z})$.

Orbifolds

- Groupoid: an étale proper Lie groupoid $G \rightrightarrows M$.
- Diffeological space: locally diffeomorphic to quotients of $\mathbb{R}^n$ by finite group actions.

Fact: every étale proper Lie groupoid is locally isomorphic to action groupoids of finite linear actions on $\mathbb{R}^n$.

Groupoids

Definitions

An n-**quasifold groupoid** is a Lie groupoid $G \rightrightarrows G_0$ for which G is Hausdorff, and about each $x \in G_0$, there is

- a neighbourhood U,
- a countable group Γ acting affinely on $\mathbb{R}^n$,
- an open subset $V \subseteq \mathbb{R}^n$,
- and a Lie groupoid isomorphism (chart) $G|_U \xrightarrow{\cong} (\Gamma \ltimes \mathbb{R}^n)|_V$

An étale Lie groupoid (i.e. $\dim G = \dim G_0$) is **effective** if it the correspondence

$$g \mapsto \mathrm{germ}_{s(g)}\, t \circ \sigma$$

is one-to-one, where σ is a section of s through g. For a quasifold groupoid G, if the actions of the groups Γ are effective, then G is effective.

Notes and Examples

- Every groupoid in Motivation is a quasifold groupoid.
- We may define an **orbifold groupoid** as a quasifold groupoid where the groups Γ are finite. It need not be proper.

Morita equivalence

Lie groupoids G and H are **Morita equivalent** if there is an invertible bibundle between them. For action groupoids $G \ltimes G_0$ and $H_0 \rtimes H$, an invertible bibundle is:

$$\begin{array}{ccccc} G \circlearrowright & & Q & & \circlearrowleft H \\ & \swarrow^{\pi'} & & \searrow^{\pi} & \\ G \circlearrowright G_0 & & & & H_0 \circlearrowleft H \end{array}$$

where

$$Q \xrightarrow{\pi'} G_0 \text{ is } H\text{-principal, } G\text{-equivariant, and } Q \xrightarrow{\pi} H_0 \text{ is } G\text{-principal, } H\text{-equivariant.}$$

- Lie groupoids, bibundles, and morphisms of bibundles, form a bicategory.
- Lie groupoids, and isomorphism classes of bibundles, form the (Hilsum-Skandalis) category.

An Equivalence of Categories

The quotient functor $\mathbf{F}$ assigns:

to a Lie groupoid $G \rightrightarrows G_0$ its diffeological orbit space G_0/G
to a bibundle $G \to H$ a smooth function $G_0/G \to H_0/H$.

It sends quasifold groupoids to diffeological quasifolds, and Morita equivalences to diffeomorphisms.

Theorem. $\mathbf{F}$ restricts to an equivalence of the categories:

(effective quasifold groupoids, with (isomorphism classes) of locally invertible bibundles), and (diffeological quasifolds, with local diffeomorphisms).

Diffeology

Definitions

- A **diffeological space** is a set X equipped with a **diffeology** – a set of maps from open subsets of Cartesian spaces into X, called *plots*, s.t.
  - (concreteness) constant maps are plots;
  - (presheaf) given $\mathscr{V} \xrightarrow{F} \mathscr{U} \xrightarrow{p} X$, if p is a plot, so is F^*p;
  - (sheaf) if $p : \mathscr{U} \to X$ is a map, and about every $r \in \mathscr{U}$, there is a neighbourhood $\mathscr{V}$ such that $p|_{\mathscr{V}} : \mathscr{V} \to X$ is a plot, then p is a plot.
- The **subset diffeology** on $A \subseteq X$ is

$$\{\text{plots of } X \text{ with image contained in } A\}.$$

- For a equivalence relation $\mathcal{R}$ on X, the **quotient diffeology** on $X/\mathcal{R}$ is

$$\{p : \mathscr{U} \to X/\mathcal{R} \mid p \text{ has local lifts everywhere}\}, \text{ i.e.}$$

$$\forall r \in \mathcal{U}, \quad \begin{array}{ccccc} & & & & X \\ & & & \nearrow^{\exists q} & \downarrow \pi \\ \exists(\mathscr{V} \ni r) & \hookrightarrow & \mathscr{U} & \xrightarrow{p} & X/\mathcal{R}. \end{array}$$

- A map $f : X \to Y$ is **smooth** if $p^* f$ is smooth for all plots p of X.

Iglesias-Zemmour and Prato, 2020: An n-**diffeological quasifold** is a diffeological space that is locally diffeomorphic to quotients of $\mathbb{R}^n$ by countable affine actions.

Notes and (non-)Examples

- Every diffeological space in Motivation is a diffeological quasifold.
- We construct a foliation whose leaf space is not a quasifold.

Proof Outline

Lemma. (Iglesias-Zemmour and Prato, 2020) If $h : \mathbb{R}^n \to \mathbb{R}^n$ is smooth, and preserves the orbits of a countable group Γ acting affinely and effectively, then $h = \gamma$ for some unique $\gamma \in \Gamma$.

Proof: uses the Baire category theorem. Then, we prove:

- An effective quasifold groupoid G is isomorphic to $\Gamma^G \rightrightarrows G_0$, where

$$\Gamma^G = \{\mathrm{germ}_x \psi \mid \psi : G_0 \to G_0 \text{ local diffeo. preserving } G\text{-orbits}\}$$

- If $f : G_0/G \to H_0/H$ is a diffeo., an invertible bibundle $\Gamma^G \to \Gamma^H$ is

$$\{\mathrm{germ}_y \psi \mid \psi : U'_\alpha \to U_i \text{ local diffeo. and } f\pi\psi = \pi'\},$$

where the U_i and U'_α are charts for G and H, respectively.

References

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