Shifted Contact Structures on Differentiable Stacks

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Shifted Symplectic Structures [1, 3, 2]

Let $G \rightrightarrows M$ be a Lie groupoid, $G^\bullet$ its nerve and $[M/G]$ the differentiable stack presented by G.

Symplicial Cohomology in Differential Forms:

$$0 \longrightarrow \Omega^\bullet(M) \xrightarrow{\partial} \Omega^\bullet(G) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega^\bullet(G^k) \xrightarrow{\partial} \cdots$$

∂ is the alternating sum of the pull-backs along the face maps of $G^\bullet$. *Multiplicative forms* on G are those in $\Omega^\bullet_{mult}(G) := \ker\left(\partial : \Omega^\bullet(G) \to \Omega^\bullet(G^2)\right)$.

Definition (Shifted Symplectic Structure)

A $+1$-*shifted* 2-*form* on $[M/G]$ is the symplicial cohomology class $[\omega]$ of a multiplicative 2-form $\omega \in \Omega^2_{mult}(G)$. $[\omega]$ is *symplectic* if it is both:

- *non-degenerate*, i.e. $\omega_\flat : TG \to T^*G$ is a Morita map, and
- *closed*, i.e. $d\omega = 0$ mod cohomologically trivial terms wrt the simplicial cohomology.

Contact Structures

A *contact structure* on a manifold N is a hyperplane distribution $K \subseteq TN$ which is *maximally non-integrable*, i.e. the *curvature*

$$R_K : \wedge^2 K \to L := TM/K, \quad (X, Y) \mapsto [X, Y] \operatorname{mod} K$$

is non-degenerate.

A contact structure can be equivalently described as a *contact form*:

more appropriate for our purposes

Definition (Contact Form)

A *contact form* is a non-zero line bundle valued 1-form $\theta \in \Omega^1(M, L)$ such that

- $R_\theta := (d^\nabla\theta)|_K : \wedge^2 K \to L$ is non-degenerate ($K := \ker\theta$),

for one, hence for any, connection ∇ in L (notice that $R_\theta = R_K$).

Atiyah Forms [4]

Let $L \to N$ be a line bundle and $\mathrm{At}(L) \Rightarrow N$ its *gauge algebroid*. The anchor $\sigma : \mathrm{At}(L) \to TN$ is the *symbol*. *Atiyah forms* on L are L-valued cochains $\omega : \wedge^\bullet \mathrm{At}(L) \to L$, that we denote $\Omega^\bullet_{\mathrm{At}}(L)$. Atiyah forms come with a Lie algebroid *differential*

$$d_{\mathrm{At}} : \Omega^\bullet_{\mathrm{At}}(L) \to \Omega^{\bullet+1}_{\mathrm{At}}(L).$$

Components of an Atiyah Form:

There is a short exact sequence of vector spaces

$$0 \longrightarrow \Omega^\bullet(N, L) \xrightarrow{\sigma^*} \Omega^\bullet_{\mathrm{At}}(L) \longrightarrow \Omega^{\bullet-1}(N, L) \longrightarrow 0$$

splitting via

$$\Omega^{\bullet-1}(N, L) \to \Omega^\bullet_{\mathrm{At}}(L), \quad \theta \mapsto d_{\mathrm{At}}(\sigma^*\theta).$$

Accordingly, there is a vector space decomposition

$$\Omega^\bullet_{\mathrm{At}}(L) \cong \Omega^{\bullet-1}(N, L) \oplus \Omega^\bullet(N, L), \quad \omega \rightleftharpoons (\omega_0, \omega_1).$$

Symplectic Atiyah Forms

symplectic-like approach to contact structures

An Atiyah 2-form $\omega \in \Omega^2_{\mathrm{At}}(L)$ is *symplectic* if it is both:

- *non-degenerate*, i.e. $\omega_\flat : \mathrm{At}(L) \to \mathrm{At}(L)^* \otimes L$ is an isomorphism, and
- *closed*, i.e. $d_{\mathrm{At}}\omega = 0$, equivalently $\omega_1 = 0$.

Proposition (Contact Structures Revisited)

The assignment $\theta \mapsto \omega \rightleftharpoons (\theta, 0)$ establishes a bijection between

- contact 1-forms and
- symplectic Atiyah forms.

LB Groupoids

line bundles in the category of groupoids

Let $(L \rightrightarrows L_M) \to (G \rightrightarrows M)$ be a line bundle (LB) groupoid (all the structure maps are isomorphisms on fibers), let $L^\bullet \to G^\bullet$ be its nerve and $[L_M/L] \to [M/G]$ the vector bundle in differentiable stacks presented by L.

Symplicial Cohomology in L-Valued Forms:

$$0 \longrightarrow \Omega^\bullet(M, L_M) \xrightarrow{\partial} \Omega^\bullet(G, L) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega^\bullet(G^k, L^k) \xrightarrow{\partial} \cdots$$

∂ is the alternating sum of the pull-backs along the face maps of $L^\bullet$. *Multiplicative L-valued forms* on G are those in $\Omega^\bullet_{mult}(G, L) := \ker\left(\partial : \Omega^\bullet(G, L) \to \Omega^\bullet(G^2, L^2)\right)$.

Shifted Contact Structures

Let $\theta \in \Omega^1_{mult}(G, L)$. We can always make sense of $\ker\theta$ as a differentiable stack:

Theorem (Morita Kernel & Morita Curvature)

(1) There is a canonical VB groupoid structure

$$\begin{tikzcd} MK_\theta \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & TM \oplus L_M \arrow[d] \\ G \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

inspired by the *homotopy kernel* in homological algebra

on $MK_\theta := TG \oplus L$ with target

$$\tilde{t}(v, \ell) := \left(dt(v), t^L(\ell + \theta(v))\right).$$

There is also an *adjoint VB groupoid* $MK^\dagger_\theta := (T^*G \otimes L) \oplus \mathbb{R}_M \rightrightarrows A^* \otimes L_M$.

(2) For any connection ∇ in L_M, let $\eta_\nabla := s^*\nabla - t^*\nabla$ ($\in \Omega^1_{mult}(G)$). Then, the VB map:

$$MR_\theta = \begin{pmatrix} d^{t^*\nabla}\theta & \eta_\nabla \\ -\eta_\nabla & 0 \end{pmatrix} : MK_\theta \to MK^\dagger_\theta$$

is a VB groupoid map.

(3) MK_θ and MR_θ do only depend on $[L_M/L]$ and the simplicial cohomology class $[\theta]$ up to Morita equivalences.

Motivating Remark:

When θ is non-zero, the inclusion $K := \ker\theta \hookrightarrow MK_\theta$ is Morita.

Definition (Shifted Contact Form)

A $+1$-*shifted contact form* on $[M/G]$ is the symplicial cohomology class $[\theta]$ of a multiplicative 1-form $\theta \in \Omega^1(G, L)$ such that

- $MR_\theta : MK_\theta \to MK^\dagger_\theta$ is a Morita map.

Multiplicative Atiyah Forms

Let $(L \to G) \rightrightarrows (L_M \to M)$ be an LB groupoid.

Symplicial Cohomology in Atiyah Forms:

$$0 \longrightarrow \Omega^\bullet_{\mathrm{At}}(L_M) \xrightarrow{\partial} \Omega^\bullet_{\mathrm{At}}(L) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega^\bullet_{\mathrm{At}}(L^k) \xrightarrow{\partial} \cdots$$

Multiplicative Atiyah forms are those in $\Omega^\bullet_{\mathrm{At},mult}(L) := \ker\left(\partial : \Omega^\bullet_{\mathrm{At}}(L) \to \Omega^\bullet_{\mathrm{At}}(L^2)\right)$.

$\omega \in \Omega^\bullet_{\mathrm{At}}(L)$ is multiplicative iff ω_0, ω_1 are so.

Shifted Symplectic Atiyah Forms

Definition (Shifted Symplectic Atiyah Form)

A $+1$-*shifted Atiyah 2-form* on $[L_M/L]$ is the simplicial cohomology class $[\omega]$ of a multiplicative Atiyah 2-form $\omega \in \Omega^2_{\mathrm{At},mult}(L)$. $[\omega]$ is *symplectic* if it is both:

- *non-degenerate*, i.e. $\omega_\flat : \mathrm{At}(L) \to \mathrm{At}(L)^* \otimes L$ is a Morita map, and
- *closed*, i.e. $d_{\mathrm{At}}\omega = 0$ mod cohomologically trivial terms wrt the simplicial cohomology.

Theorem (Shifted Contact & Shifted Symplectic Atiyah Forms)

The assignment $\theta \mapsto \omega \rightleftharpoons (\theta, 0)$ establishes a bijection between

- $+1$-shifted contact forms and
- $+1$-shifted symplectic Atiyah forms.

References

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