Poisson structures on sets of Maurer-Cartan elements

Thomas Machon, *University of Bristol*

University of BRISTOL

Abstract: Given a differential graded Lie algebra satisfying certain conditions, we construct Poisson structures on the gauge orbits on its set of Maurer-Cartan (MC) elements. The construction associates a Batalin-Vilkovisky algebra to each MC element and is reminiscient of the Poisson structure on the dual to a Lie algebra.

MCP structures

Definition:
A Maurer-Cartan Poisson structure is a triple $(\mathcal{L}, \mathcal{B}, (\cdot,\cdot))$

With:

$\mathcal{L} = (L, d, [\cdot,\cdot])$	differential graded Lie algebra
$\mathcal{B} = (B, \wedge)$	graded commutative algebra
$(\cdot,\cdot) : L^i \times B^{i+1} \to k$	non-degenerate pairing

Such that, for each $X \in \mathrm{MC}(\mathcal{L})$, the set of Maurer-Cartan elements, we have:

(1): $X \in \mathrm{MC}(\mathcal{L}) \implies \lambda X \in \mathrm{MC}(\mathcal{L})$

(2): The adjoint operator $\delta_X : B^i \to B^{i-1}$ defined by

$$(dY + [X, Y], \alpha) = (Y, \delta_X \alpha)$$

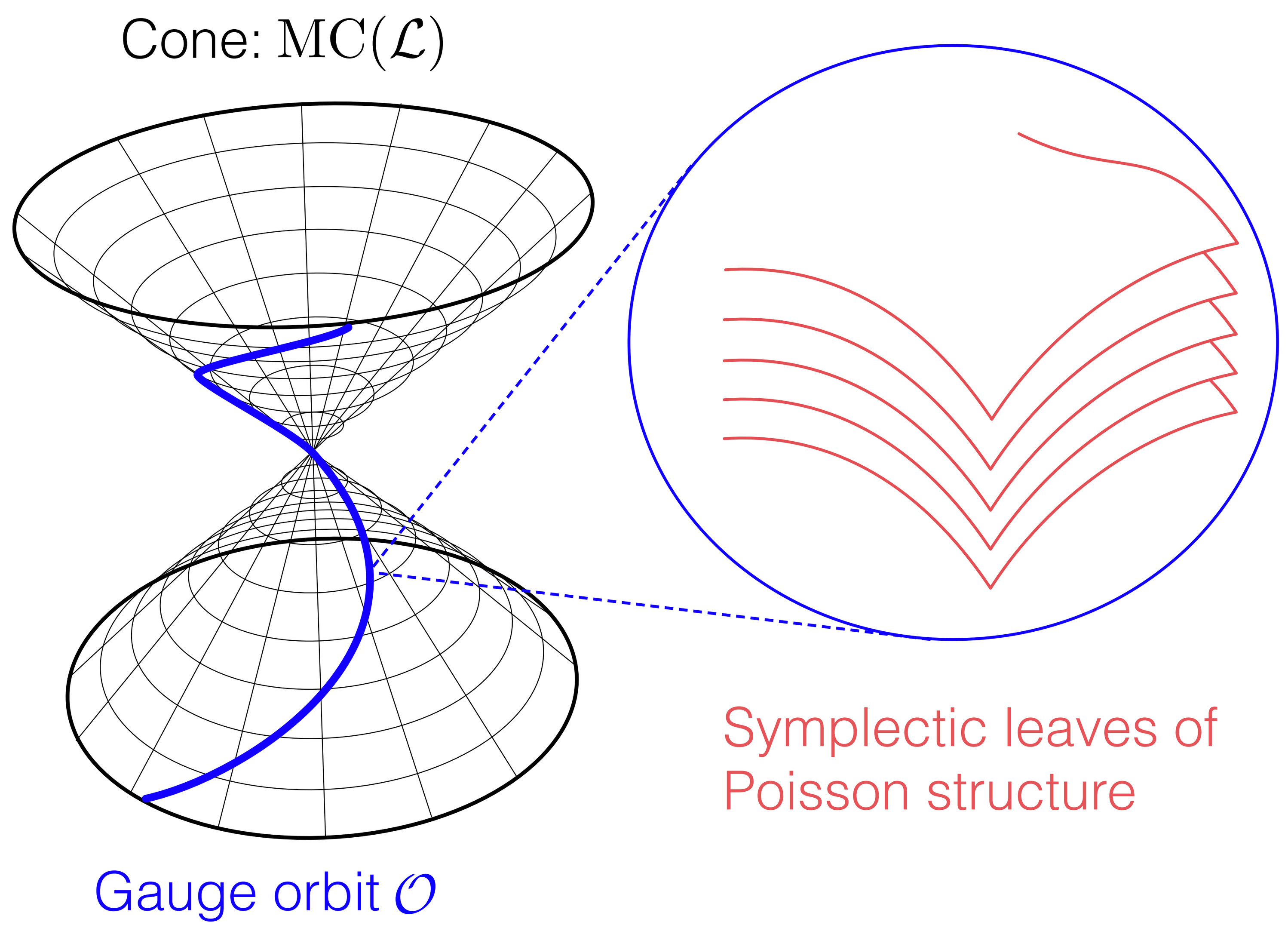
Gives $\mathcal{B}$ the structure of a Batalin-Vilkovisky algebra

(3): The map $\rho_X : B^1 \to L^0$, defined by

$$(X, \alpha \wedge \beta) = (\rho_X \alpha, \beta)$$

Is a homomorphism* of Lie algebras, where B^1 is equipped with the Gerstenhaber Lie bracket

Sketch of construction



Poisson structures on gauge orbits

The set $\mathrm{MC}(\mathcal{L})$ defined as

$$\mathrm{MC}(\mathcal{L}) = \left\{ X \in L^1 \;\middle|\; dX + \frac{1}{2}[X, X] = 0 \right\}$$

has an action of the Lie algebra L^0, decomposing it into gauge orbits, immersed submanifolds of L^1.

For a given X in an orbit $\mathcal{O}$ we have a map

$$d_X \rho_X \delta_X : T^*_X\mathcal{O} \to T_X\mathcal{O}$$

where $d_X = d + [X, \cdot]$

Theorem: this map defines a Poisson structure on $\mathcal{O}$

Poisson tensor Π depends cubically on X

$$\Pi : T^*_X\mathcal{O} \times T^*_X\mathcal{O} \to k$$
$$\Pi : (\alpha, \beta) \mapsto (X, \delta_X\alpha \wedge \delta_X\beta)$$

(note similarity to Poisson structure on dual to Lie algebra)

Example: Poisson / symplectic structures

Let (M, ω) be a closed symplectic manifold with symplectic volume form μ. Then one can construct a dgla for deformations of ω such that each gauge orbit is the quotient of (identity component of) volume preserving diffeomorphisms by symplectomorphisms

$$\mathcal{O}_\omega = \mathrm{Diff}_\mu(M)/\mathrm{Symp}(\omega)$$

Theorem: $\mathcal{O}_\omega$ has a Poisson structure

If ω satisfies the hard Lefschetz condition, this Poisson structure is non-degenerate (symplectic)†. Cotangent space of gauge orbit: $T^*_\omega\mathcal{O} = \Omega^2(M)/(\ker d^\Lambda + \mathrm{Im}\ d)$ where d^Λ is symplectic adjoint of exterior derivative. Poisson tensor then given as

$$\Pi(\alpha, \beta) = \int_M d^\Lambda\alpha \wedge d^\Lambda\beta \wedge \frac{\omega^{n-1}}{(n-1)!}$$

Isotropy lie algebra from associated Poisson Lie algebroid yields a finite-dimensional two-step nilpotent graded Lie algebra associated to (M, ω). A more general version of this construction yields a Poisson structure on the space of Posson structures.

Example: commutative Frobenius algebras

Let A be a commutative Frobenius algebra over either $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{P}(A)$ be the set of all Lie algebras on A making A a Poisson algebra. Assume finite-dimensionality.

MCP structure for this system given in terms of Hochschild (co)homology. $\mathcal{P}(A)$ is then a cone in the second Hochschild cohomology group of A, $\mathrm{HH}^2(A, A)$, defined by the vanishing of homogeneous quadratic polynomials, and is decomposed into gauge orbits.

Theorem: the coordinate ring of $\mathcal{P}(A)$ has the structure of a Poisson algebra, and each gauge orbit is a Poisson manifold.

References

T. Machon, *A Poisson bracket on the space of Poisson structures*. Accepted, J. Symplectic. Geom. arXiv:2008.11074

T. Machon,Poisson structures on sets of Maurer-Cartan elements. arXiv:2203.02310

Notes

*: in some cases one requires the existence of an element k, such that $\rho_X[a, b]_{\mathcal{B}} = k[\rho_X a, \rho_X b]$

†: in general the non-degeneracy is given in terms of Tseng and Yau's symplectic cohomology groups .