Symmetries of Singular Foliations through Universal Lie ∞-algebroids

Ruben LOUIS
ruben.louis@univ-lorraine.fr

Abstract

We associate to any Lie-Rinehart algebra over a unital commutative algebra $\mathcal{O}$, for instance to any singular foliation $\mathcal{F} \subseteq \mathfrak{X}(M)$ on a manifold M, a "unique" class of negatively graded Lie ∞-algebroids $\simeq NQ$-manifolds. [1][2]

We show that a weak symmetry action of a Lie algebra $\mathfrak{g}$ on $\mathcal{F}$ induces a unique (up to homotopy) Lie ∞-morphism from $\mathfrak{g}$ to the DGLA made of vector fields on a universal NQ-manifold of $\mathcal{F}$. [3]

Background

Definition.
A dg-manifold or NQ-manifold is a positively graded manifold $(M, E, \mathcal{E})$ endowed with a degree $+1$ homological vector field[a] Q on E, i.e., $Q \in \mathfrak{X}_1(E)$ is such that $Q^2 = 0$.

Proposition. (T. Voronov, N. Poncin,...)
There is a one-to-one correspondence between NQ-manifolds and negatively graded Lie ∞-algebroids over M.

[a] A vector field on E is a graded derivation of the algebra of functions $\mathcal{E}$. We denote by $\mathfrak{X}_\bullet(E) := \mathrm{Der}_\bullet(\mathcal{E})$ the graded Lie algebra of vector fields on E.

Weak symmetry actions

Definition.

- A weak symmetry action of a Lie algebra $\mathfrak{g}$ on a singular foliation $\mathcal{F}$ on M is a $\mathbb{K}$-linear map $\varrho\colon \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ that satisfies:
  - $\forall\, x \in \mathfrak{g},\ [\varrho(x), \mathcal{F}] \subseteq \mathcal{F}$,
  - $\forall\, x, y \in \mathfrak{g},\ \varrho([x,y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)] \in \mathcal{F}$.
- Two weak symmetry actions, $\varrho, \varrho' \colon \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ are said to be equivalent if there exists a linear map $\varphi\colon \mathfrak{g} \longrightarrow \mathcal{F}$ such that $\varrho - \varrho' = \varphi$.

Example. Let $W \subseteq \mathbb{C}^d$ be an affine variety and $\mathcal{I}_W \subset \mathbb{C}[x_1, \ldots, x_d]$ its corresponding ideal. Let $\mathcal{F}_W := \mathcal{I}_W \mathfrak{X}(\mathbb{C}^d)$ be the singular foliation made of vector fields vanishing on W. Any Lie algebra morphism $\varrho\colon \mathfrak{g} \longrightarrow \mathfrak{X}(W)$ extends to a weak symmetry action of $\mathfrak{g}$ on $\mathcal{F}_W$ over the ambient space $\mathbb{C}^d$. Any two such extensions are equivalent.

A first result on Lie-Rinehart algebras, e.g singular foliations

Theorem 1. [1][2]
Let $\mathcal{F}$ be a Lie Rinehart algebra over an algebra $\mathcal{O}$.

1. Any resolution of $\mathcal{F}$ by free $\mathcal{O}$-modules
$$\cdots \xrightarrow{\mathrm{d}} P_{-3} \xrightarrow{\mathrm{d}} P_{-2} \xrightarrow{\mathrm{d}} P_{-1} \xrightarrow{\rho} \mathcal{F} \longrightarrow 0$$
carries a Lie ∞-algebroid structure over $\mathcal{F}$ whose unary bracket is $\ell_1 := \mathrm{d}$.
2. Any two of such structures are unique up to homotopy equivalence.
3. In particular, when $\mathcal{F}$ is a singular foliation that admits a geometric resolution[a] (E, d, ρ), there exists a NQ-manifold (E, Q) over $\mathcal{F}$ called "universal", whose linear part is (E, d, ρ).

[a] i.e., $P_{-i} = \Gamma(E_{-i})$, for some vector bundle E_{-i} over M.

Application to symmetries of singular foliations

Theorem 2. [3]
Let $\mathcal{F}$ a be singular foliation over a smooth manifold (or an affine variety) M and $\varrho\colon \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ be a weak symmetry action of $\mathfrak{g}$ on $\mathcal{F}$.

1. For any universal Lie ∞-algebroid (E, Q) of $\mathcal{F}$, ϱ lifts to a Lie ∞-morphism of DGLAs
$$\Phi_\bullet\colon (\mathfrak{g}[1], [\cdot\,,\cdot]_{\mathfrak{g}}) \overset{\infty}{\rightsquigarrow} (\mathfrak{X}_\bullet(E)[1], [\cdot\,,\cdot], \mathrm{ad}_Q)\,.$$
2. Any two such lifts are homotopy equivalent.
3. Any two such lifts of any two equivalent weak symmetry actions of $\mathfrak{g}$ on $\mathcal{F}$ are homotopy equivalent.

Remark. The two first terms of the Lie-∞-morphism read

- $\Phi_0\colon \mathfrak{g} \to \mathfrak{X}_0(E)$ is such that $\Phi_0(x)[f] = \varrho(x)[f]$, and $[Q, \Phi_0(x)] = 0, \quad \forall x \in \mathfrak{g}, f \in C^\infty(M)$. This morphism is not a graded Lie algebra morphism,
- but there exists a linear map $\Phi_1\colon \wedge^2 \mathfrak{g} \to \mathfrak{X}_{-1}(E)$ such that for all $x, y \in \mathfrak{g}$: $\Phi_0([x,y]_{\mathfrak{g}}) - [\Phi_0(x), \Phi_0(y)] = [Q, \Phi_1(x,y)]$.

An obstruction theory

Proposition. [3] Let $m \in M$ be a point of M. Assume that the isotropy Lie algebra $\mathfrak{g}_m$ of $\mathcal{F}$ at m is Abelian. Then, for any weak symmetry action ϱ of $\mathfrak{g}$ on $\mathcal{F}$ such that
$$\varrho([x,y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)] = \rho(\eta(x,y)) \in \mathcal{F}(m), \quad \text{for all } x, y \in \mathfrak{g}$$

1. $\mathfrak{g}_m$ is a $\mathfrak{g}$-module.
2. The bilinear map, $\eta_{|_m}\colon \wedge^2 \mathfrak{g} \to \mathfrak{g}_m$, is a Chevalley-Eilenberg 2-cocycle.
3. Its class $\underline{cl}(\eta) \in H^2(\mathfrak{g}, \mathfrak{g}_m)$ does not depend on the choices made in the construction.
4. Furthermore, $cl(\eta)$ obstructs the existence of having a strict symmetry action equivalent to ϱ.

References

[1] Camille Laurent-Gengoux, Sylvain Lavau, and Thomas Strobl. *The universal Lie ∞-algebroid of a singular foliation.* **Doc. Math.**, 2020.

[2] Camille Laurent-Gengoux and Ruben Louis. *Lie-Rinehart algebras $\simeq$ acyclic Lie ∞-algebroids.* **Journal of algebra**, 2022.

[3] Ruben Louis. *On symmetries of singular foliations.* 2022. https://**arxiv**.org/abs/2203.01585