Homotopy momentum sections and sigma models

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Introduction

We introduce a notion of a homotopy momentum section on a Lie algebroid over a pre-multisymplectic manifold. A homotopy momentum section is a generalization of a momentum map and a momentum section on a pre-symplectic manifold, and is also a generalization of a homotopy momentum map on a multisymplectic manifold. A gauged nonlinear sigma model with Wess-Zumino term with Lie algebroid gauging has this structure.

'Homotopy momentum sections on multisymplectic manifolds,' Y. Hirota and NI, arXiv: 2110.12305 [math.SG]

Lie algebroids and differentials

Definition 1.1 *A Lie algebroid $(E, \rho, [-,-])$ is a vector bundle E over a smooth manifold M with a bundle map $\rho : E \to TM$ called the anchor map, and a Lie bracket $[-,-] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ satisfying the Leibniz rule,*

$$[e_1, f e_2] = f[e_1, e_2] + \rho(e_1)f \cdot e_2,$$

for $e_i \in \Gamma(E)$ and $f \in C^\infty(M)$.

Definition 1.2 *A Lie algebroid differential ${}^E\mathrm{d} : \Gamma(\wedge^m E^*) \to \Gamma(\wedge^{m+1} E^*)$ is defined by*

$$\begin{aligned}
&{}^E\mathrm{d}\alpha(e_1, \ldots, e_{m+1}) \\
&= \sum_{i=1}^{m+1} (-1)^{i-1} \rho(e_i)\alpha(e_1, \ldots, \check{e}_i, \ldots, e_{m+1}) \\
&+ \sum_{1 \le i < j \le m+1} (-1)^{i+j} \alpha([e_i, e_j], e_1, \ldots, \check{e}_i, \ldots, \check{e}_j, \ldots, e_{m+1}),
\end{aligned}$$

where $\alpha \in \Gamma(\wedge^m E^*)$ and $e_i \in \Gamma(E)$.

We introduce an ordinary connection on the vector bundle E, $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$, which is a $\mathbb{R}$-linear map satisfying the Leibniz rule, $\nabla(fe) = f\nabla e + (\mathrm{d}f) \otimes e$ for $e \in \Gamma(E)$ and $f \in C^\infty(M)$.

Definition 1.3 *An E-connection on a vector bundle E' with respect to a Lie algebroid E is a $\mathbb{R}$-linear map ${}^E\nabla : \Gamma(E') \to \Gamma(E^* \otimes E')$ satisfying*

$$ {}^E\nabla_e(fe') = f\,{}^E\nabla_e e' + (\rho(e)f)e', \qquad (1)$$

for $e \in \Gamma(E)$, $e' \in \Gamma(E')$ and $f \in C^\infty(M)$.

If a normal connection ∇ on E is given, a (canonical) E-connection on a tangent bundle, ${}^E\nabla : \Gamma(TM) \to \Gamma(E^* \otimes TM)$ is defined by

$$ {}^E\nabla_e v := \mathcal{L}_{\rho(e)} v + \rho(\nabla_v e) = [\rho(e), v] + \rho(\nabla_v e),$$

where $e \in \Gamma(E)$ and $v \in \mathfrak{X}(M)$.

Definition 1.4 *For $\Omega^k(M, \wedge^m E^*) = \Gamma(\wedge^k T^*M \otimes \wedge^m E^*)$, the E-exterior covariant derivative ${}^E\mathrm{d}^\nabla : \Omega^k(M, \wedge^m E^*) \to \Omega^k(M, \wedge^{m+1} E^*)$ is defined by*

$$\begin{aligned}
&{}^E\mathrm{d}^\nabla \alpha(e_1, \ldots, e_{m+1}) \qquad (2)\\
&:= \sum_{i=1}^{m+1} (-1)^{i-1}\, {}^E\nabla_{e_i}(\alpha(e_1, \ldots, \check{e}_i, \ldots, e_{m+1})) \\
&+ \sum_{1 \le i < j \le m+1} (-1)^{i+j} \alpha([e_i, e_j], e_1, \ldots, \check{e}_i, \ldots, \check{e}_j, \ldots, e_{m+1}),
\end{aligned}$$

for $\alpha \in \Omega^k(M, \wedge^m E^*)$ and $e_i \in \Gamma(E)$.

Homotopy momentum sections

Definition 1.5 *A pre-n-plectic manifold is a pair (M, ω), where M is a smooth manifold and ω is a closed $(n+1)$-form on M.*

We introduce a series of k-forms taking values in $\wedge^{n-k} E^*$, $\mu_k \in \Omega^k(M, \wedge^{n-k} E^*)$, where $k = 0, \ldots, n-1$.

Definition 1.6 *A formal sum $\mu = \sum_{k=0}^{n-1} \mu_k$ is called a homotopy momentum section if μ satisfies*

$$(\nabla + {}^E\mathrm{d}^\nabla)\mu = -\sum_{k=0}^{n} \iota_\rho^{n+1-k} \omega. \qquad (3)$$

Here, $\iota_\rho^k \omega \in \Omega^{n-k}(M, \wedge^k E^*)$ is given by

$$\begin{aligned}
&\iota_\rho^k \omega(v_{k+1}, \ldots, v_{n+1})(e_1, \ldots, e_k) \\
&:= \omega(\rho(e_k), \ldots, \rho(e_1), v_{k+1}, \ldots, v_{n+1}),
\end{aligned}$$

for $e_1, \ldots, e_k \in \Gamma(E)$ and $v_{k+1}, \ldots, v_{n+1} \in \mathfrak{X}(M)$.

Expanding the equation by form degree, Equation (3) is the following $n+1$ equations,

$$\begin{aligned}
&\nabla\mu_{n-1} = -\iota_\rho^1 \omega, \\
&\nabla\mu_{k-1} + {}^E\mathrm{d}^\nabla \mu_k = -\iota_\rho^{n+1-k} \omega, \quad (k = 1, \ldots, n-1) \\
&{}^E\mathrm{d}^\nabla \mu_0 = {}^E\mathrm{d}\mu_0 = -\iota_\rho^{n+1} \omega.
\end{aligned}$$

Example 1.7 (Momentum map on symplectic manifold) *The Lie group action on M induces an action Lie algebroid structure on a trivial bundle $E = M \times \mathfrak{g}$ with a Lie algebra $\mathfrak{g}$ of G. $\nabla = \mathrm{d}$ with the de Rham differential d.*

μ has only one component $\mu = \mu_0 \in C^\infty(M, \mathfrak{g}^*)$ which is regarded as $\mu : M \to \mathfrak{g}^*$. Equation (3) reduces to two equations,

$$\mathrm{d}\mu_0 = -\iota_\rho \omega, \qquad {}^E\mathrm{d}\mu_0 = -\iota_\rho^2 \omega.$$

It is equivalent to the condition of the momentum map,

$$\mathrm{d}\mu_0 = -\iota_\rho \omega, \qquad \mu_0([e_1, e_2]) = \mathrm{ad}^*_{e_1} \mu_0(e_2).$$

for $e_1, e_2 \in \mathfrak{g}$.

Example 1.8 (Momentum map on multisymp. manifold) *Let $n \ge 2$ and (M, ω) be an n-plectic manifold with an n-plectic form ω.*

If we set $\mu_k = 0$ for $k = 0, \ldots, n-2$, μ has only one component, an $n-1$ form $\mu = \mu_{n-1} \in \Omega^{n-1}(M, \mathfrak{g}^*)$. Equation reduces to two equations,

$$\mathrm{d}\mu_{n-1} = -\iota_\rho \omega, \qquad \mu_{n-1}([e_1, e_2]) = \mathrm{ad}^*_{e_1} \mu_{n-1}(e_2).$$

for $e_1, e_2 \in \mathfrak{g}$.

Example 1.9 (Momentum section on sympl. manifold) *Take $n = 1$ and a general Lie algebroid E. M is a pre-symplectic manifold. Similar to the momentum map, a homotopy momentum section μ is only one component $\mu = \mu_0 \in \Gamma(E^*)$. Equation (3) reduces to two equations,*

$$\begin{aligned}
&\nabla\mu_0 = -\iota_\rho \omega, \\
&{}^E\mathrm{d}\mu_0 = -\iota_\rho^2 \omega.
\end{aligned}$$

These equations are that μ_0 is a momentum section on a Lie algebroid over a pre-symplectic manifold introduced by Blohmann and Weinstein.

Homotopy momentum map

Let (M, ω) be an n-plectic manifold. Suppose an action of a Lie group G on M. The action of G induces the corresponding infinitesimal Lie algebra action on M as vector fields, $\rho : \mathfrak{g} \to TM$. We have two differentials on the space, $\Omega^k(M, \wedge^{n-k}\mathfrak{g}^*) = \Omega^k(M) \otimes \wedge^{n-k}\mathfrak{g}^*$, $\mathrm{d} := \mathrm{d} \otimes 1$ and $\mathrm{d}_{CE} := 1 \otimes \mathrm{d}_{CE}$ where d is the de Rham differential on $\Omega^k(M)$ and d_{CE} is the Chevalley-Eilenberg differential on $\wedge^{n-k}\mathfrak{g}^*$,

$$\begin{aligned}
&\mathrm{d}_{CE}\alpha(e_1, \ldots, e_{m+1}) \\
&= \sum_{1 \le i < j \le m+1} (-1)^{i+j} \alpha([e_i, e_j], e_1, \ldots, \check{e}_i, \ldots, \check{e}_j, \ldots, e_{m+1}).
\end{aligned}$$

Introduce $\widehat{\mu} = \sum_{k=0}^{n-1} \widehat{\mu}_k$, where $\widehat{\mu}_k \in \Omega^k(M, \wedge^{n-k}\mathfrak{g}^*)$ is a k-form taking a value in $\wedge^{n-k}\mathfrak{g}^*$, where $k = 0, \ldots, n-1$.

Definition 1.10 *A homotopy momentum map $\widehat{\mu}$ is defined by*

$$(\mathrm{d} + \mathrm{d}_{CE})\widehat{\mu} = \sum_{k=0}^{n} (-1)^{n-k+1} \iota_\rho^{n+1-k} \omega. \qquad (4)$$

Equivariant differential forms

Definition 1.11 *$\alpha \in \Omega^l(M, \wedge^m E^*)$ is called equivariant with respect to a Lie algebroid $(E, \rho, [-,-])$ if it satisfies*

$${}^E\nabla_e \alpha(e_1, \ldots, e_m) = \sum_{i=1}^{m} (-1)^{i-1} \alpha([e, e_i], e_1, \ldots, \check{e}_i, \ldots, e_m),$$

for $e, e_i \in \Gamma(E)$.

Note: a momentum map is equivariant.

In this case, the E-exterior covariant derivative (2) becomes

$$\begin{aligned}
&{}^E\mathrm{d}^\nabla \alpha(e_1, \ldots, e_{m+1}) \qquad (5)\\
&= -\sum_{1 \le i < j \le m+1} (-1)^{i+j} \alpha([e_i, e_j], e_1, \ldots, \check{e}_i, \ldots, \check{e}_j, \ldots, e_{m+1}).
\end{aligned}$$

Let $n \ge 2$. A Lie group action induces an action Lie algebroid structure on the trivial bundle $E = M \times \mathfrak{g}$.

If the homotopy momentum section μ on the action Lie algebroid is equivariant, Equation (5) is ${}^E\mathrm{d}^\nabla = -\mathrm{d}_{CE}$. Thus, the homotopy momentum section reduces to

$$(\mathrm{d} - \mathrm{d}_{CE})\mu = -\sum_{k=0}^{n} \iota_\rho^{n+1-k} \omega.$$

If we take $\widehat{\mu}_k = (-1)^{n-k+1} \mu_k$, Equation (6) coincides with Equation (4). Therefore, we obtain the following theorem.

Theorem 1.12 *Let $n \ge 2$ and a Lie algebroid be the action Lie algebroid $E = M \times \mathfrak{g}$ with a Lie algebra $\mathfrak{g}$. Then, If the homotopy momentum section μ for the action Lie algebroid E is equivariant, it is a homotopy momentum map for $\mathfrak{g}$.*

Gauged nonlinear sigma model with Wess-Zumino term

Ξ is an $(n+1)$-dimensional smooth manifold Ξ with n-dimensional boundary $\Sigma = \partial\Xi$. M is a d-dimensional manifold. Suppose that M is a d-dimensional pre-n-plectic Riemannian manifold with a metric g. E is a Lie algebroid over M. Let $X : \Xi \to M$ be a map from Ξ to M.

We consider a *gauged* nonlinear sigma model by introducing a 'connection' D on Σ. Let $A \in \Omega^1(\Sigma, X^*E)$ be a 1-form on Σ taking a value in the pullback of E corresponding to D. Moreover, introduce a connection ∇ on E and its connection 1-form $\omega_{ai}^b(x)\mathrm{d}x^i \otimes e^a \otimes e_b$. Take local coordinate expressions of the anchor map and the Lie bracket as

$$\rho(e_a) := \rho_a^i(x)\partial_i, \qquad [e_a, e_b] := C_{ab}^c(x)e_c.$$

(Covariant) gauge transformations are

$$\begin{aligned}
&\delta X^i = \rho_a^i(X)\epsilon^a, && (6)\\
&\delta A^a = \mathrm{d}\epsilon^a + C_{bc}^a(X)A^b\epsilon^c + \omega_{bi}^a(X)\epsilon^b DX^i. && (7)
\end{aligned}$$

Here $DX^i = \mathrm{d}X^i - \rho_a^i(X)A^a$.

The action functional S is a so called Hull-Spence type,

$$S_{LA} = S_g + S_H + S_\mu, \qquad (8)$$

where

$$\begin{aligned}
S_g &= \int_\Sigma \frac{1}{2} g_{ij}(X) DX^i \wedge *DX^j \\
S_H &= \int_\Xi X^*H, \\
S_\mu &= \sum_{k=0}^{n-1} \int_\Sigma \frac{1}{k!(n-k)!} X^*\tilde{\mu}^{(k)}_{i_1 \ldots i_k a_{k+1} \ldots a_n} \mathrm{d}X^{i_1} \wedge \ldots \wedge \mathrm{d}X^{i_k} \\
&\wedge A^{a_{k+1}} \wedge \ldots \wedge A^{a_n},
\end{aligned}$$

where $X^*\tilde{\mu}^{(k)}$ is the pullback of a k-form on M taking a value in $\wedge^{n-k}E^*$, $\tilde{\mu}^{(k)} \in \Omega^k(M, \wedge^{n-k}E^*)$. Requiring (8) is invariant under gauge transformations (6) and (7), we obtain geometric conditions for a metric g, H and $\tilde{\mu}^{(k)}$.

The condition for the metric g is ${}^E\nabla g = 0$.

For H and $\tilde{\mu}^{(k)}$, define

$$\begin{aligned}
&\mu_k := (-1)^{(k+1)\ldots(n-1)} \tilde{\mu}^{(k)}, \\
&\omega(\rho(e_1), \ldots, \rho(e_k), v_{k+1}, \ldots, v_{n+1}) \\
&\quad := (-1)^{n-k} H(\rho(e_k), \ldots, \rho(e_1), v_{k+1}, \ldots, v_{n+1}),
\end{aligned}$$

where $e_1, \ldots, e_k \in \Gamma(E)$ and $v_{k+1}, \ldots, v_{n+1} \in \mathfrak{X}(M)$.

The action functional S_{LA} is gauge invariant iff μ_k satisfy the definition of the homotopy momentum section and the equation,

$$\mu_{k-1} = \iota_\rho \mu_k,$$

for $k = 1, \ldots, n-1$. We summarize our result.

Proposition 1.13 *We consider the n-dimensional nonlinear sigma model with $(n+1)$-dimensional WZ term. If we consider gauging by a Lie algebroid, the theory has a homotopy momentum section structure.*