Extending the Large Color R-matrix to the three-fold symmetric Lie algebra $\mathfrak{so}_8$

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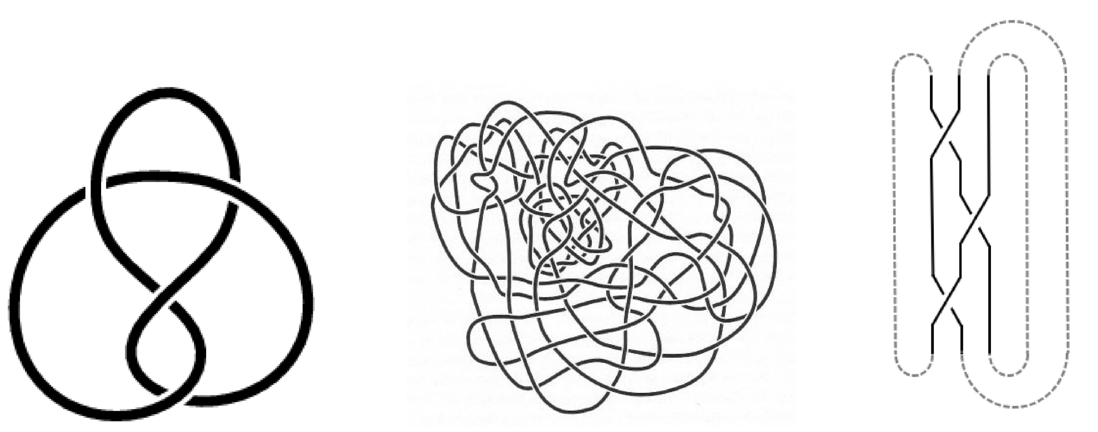
Abstract

The Gukov-Manolescu series, denoted $F_K(x,q)$, is a conjectural invariant of knot complements that, in a sense, analytically continues the colored Jones polynomials [GM19]. In a recent paper [Par20], Sunghyuk Park uses the large color R-matrix of $\mathfrak{sl}_2$ to study F_K for some simple links, giving a definition of F_K for positive braid knots and computing F_K for various knots and links. This procedure has been extended by Angus Gruen to all other Lie algebras $\mathfrak{sl}_{n+1}$ beyond $\mathfrak{sl}_2$. In this work, we turn to the family $\mathfrak{so}_{2n}$ of complex semisimple Lie algebras in Cartan's classification, mainly focusing on the $\mathfrak{so}_8$ case attracted by the three-fold symmetry in its Dynkin diagram D_4.

Braiding and Large Color R-matrix

Definition 1 A **link** is a finite collection of smoothly embedded closed curves $\mathbb{S}^1 \sqcup \cdots \sqcup \mathbb{S}^1 \hookrightarrow M$ in a closed orientable 3-manifold M. A one-component link is called a **knot**. A **knot invariant** is a quantity (e.g. a polynomial) defined on the set of all knots, which takes the same value within every isotopy class. ◂



$$\begin{array}{c} V\otimes V\otimes V \\ \uparrow R\otimes \mathrm{id}_V \\ V\otimes V\otimes V \\ \uparrow \mathrm{id}_V\otimes R \\ V\otimes V\otimes V \\ \uparrow R\otimes \mathrm{id}_V \\ V\otimes V\otimes V \end{array}$$

Proposition 2 Let V be a vector space and $R \in \mathrm{Aut}(V\otimes V)$ a solution (called an **R-matrix**) to the Yang-Baxter equation

$$(R\otimes \mathrm{id}_V)(\mathrm{id}_V\otimes R)(R\otimes \mathrm{id}_V) = (\mathrm{id}_V\otimes R)(R\otimes \mathrm{id}_V)(\mathrm{id}_V\otimes R).$$

Then, there exists a braid group representation $\rho_n : B_n \to \mathrm{Aut}(V^{\otimes n})$ such that $\rho_n(\sigma_i) = R_i$, $\forall i = 1,\dots,n-1$. Closing the braid corresponds then to taking the trace of the associated automorphism, an operation that is invariant under the Markov moves and yields thus an invariant of knots and links. ◂

Definition 3 The **Large Color R-matrix** [Par20] for a complex semisimple Lie algebra $\mathfrak{g}$ corresponds to the Yang-Baxter solutions for specific infinite-dimensional (Verma) modules of the quantum group $U_q(\mathfrak{g})$ over $\mathbb{C}(q,x)$ appearing naturally in the large color limit of the r-th symmetric representations V_r, with $x := q^r$. ◂

Remark 4 Given the Large Color R-matrix, $R(x,q)$, one can recover the Gukov-Manolescu series $F_K(x,q)$ in [GM19] by means of the quantum trace operation. ◂

General recipe

Definition 5 Let $\mathfrak{g}$ be a c.s.s. Lie algebra with Cartan matrix $A = (a_{ij})_{1\le i,j\le n}$ and root lengths $D = \mathrm{diag}(d_1,\dots,d_n)$, $d_i = \frac{1}{2}(\alpha_i,\alpha_i)$. Then, the algebra $U_q(\mathfrak{g})$ is the algebra generated by $\{X_i, Y_i, H_i\}_{1\le i,j\le n}$ and the relations

$$[H_i,H_j]=0,\quad [X_i,Y_j] = \delta_{ij}\frac{e^{hd_iH_i/2}-e^{-hd_iH_i/2}}{e^{hd_i/2}-e^{-hd_i/2}},\quad q_i = e^{hd_i/2}$$
$$[H_i,X_j] = a_{ij}X_j,\quad [H_i,Y_j] = -a_{ij}Y_j,$$
◂

and if $i\neq j$

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\begin{bmatrix}1-a_{ij}\\k\end{bmatrix}X_i^kX_jX_i^{1-a_{ij}-k}=0,\qquad \sum_{k=0}^{1-a_{ij}}(-1)^k\begin{bmatrix}1-a_{ij}\\k\end{bmatrix}Y_i^kY_jY_i^{1-a_{ij}-k}=0.$$

The **quantum integer** is defined by $[n] := \frac{q^n-q^{-n}}{q-q^{-1}} = q^{-n+1}+q^{-n+3}+\cdots+q^{n-3}+q^{n-1}$.

Theorem 6 ([KR90; Ros91]) *The universal R-matrix for $U_q(\mathfrak{g})$ is given by*

$$R = \prod_{\alpha\in\Delta_+}\left(\sum_{\ell=0}^{\infty}\frac{(1-q_\alpha^{-2})^\ell}{[\ell]_\alpha!}q_\alpha^{\frac{\ell(\ell-1)}{2}}(E_\alpha^\ell\otimes F_\alpha^\ell)\right)\exp\left(\frac{h}{2}\sum_{i,j}B_{ij}H_i\otimes H_j\right)$$

where the order on Δ_+ is given by the reduced decomposition $w_0 = r_{i_1}\cdots r_{i_\nu}$ of the longest element in the Weyl group ($r_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$ reflections). The E_α and F_α are then given by

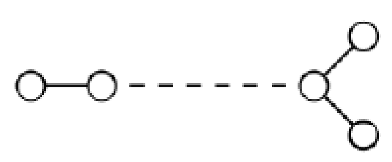
$$E_{\beta_s} = T_{i_1}\cdots T_{i_{s-1}}(X_{i_s}),\quad F_{\beta_s} = T_{i_1}\cdots T_{i_{s-1}}(Y_{i_s}),\quad s = 1,\dots,\nu,$$

with T_i specific automorphisms acting on the generators. Finally, q_α is $q^{d_i} = e^{\frac{hd_i}{2}}$ for each α in the orbit of α_i under the action of the Weyl group and $(B_{ij})_{1\le i,j\le n}$ is the inverse matrix of $DA = (d_ia_{ij})_{1\le i,j\le n}$. ◂

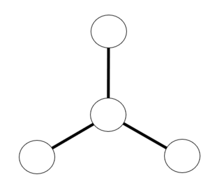
Towards the $\mathfrak{so}_8$ Large Color R-matrix

$$\mathfrak{so}(m,\mathbb{C}) = \{X\in\mathfrak{gl}(m,\mathbb{C}) \mid X + X^t = 0\}$$

First cases: $\mathfrak{so}_2\cong\mathbb{C}$, $\mathfrak{so}_4\cong\mathfrak{sl}_2\oplus\mathfrak{sl}_2$, $\mathfrak{so}_6\cong\mathfrak{sl}_4$, $\mathfrak{so}_8\rightsquigarrow D_4$.



$$A = \begin{pmatrix}2&-1&0&0\\-1&2&-1&-1\\0&-1&2&0\\0&-1&0&2\end{pmatrix}$$



Proposition 7 The r-th symmetric representations of $\mathfrak{so}_8$ are provided by considering the polynomial representation with generators

$$X_1 = i\left(x_4\frac{\partial}{\partial x_2}-x_1\frac{\partial}{\partial x_3}\right),\quad Y_1 = i\left(x_3\frac{\partial}{\partial x_1}-x_2\frac{\partial}{\partial x_4}\right)$$
$$X_2 = i\left(x_6\frac{\partial}{\partial x_4}-x_3\frac{\partial}{\partial x_5}\right),\quad Y_2 = i\left(x_5\frac{\partial}{\partial x_3}-x_4\frac{\partial}{\partial x_6}\right)$$
$$X_3 = i\left(x_8\frac{\partial}{\partial x_6}-x_5\frac{\partial}{\partial x_7}\right),\quad Y_3 = i\left(x_7\frac{\partial}{\partial x_5}-x_6\frac{\partial}{\partial x_8}\right)$$
$$X_4 = i\left(x_7\frac{\partial}{\partial x_6}-x_5\frac{\partial}{\partial x_8}\right),\quad Y_4 = i\left(x_8\frac{\partial}{\partial x_5}-x_6\frac{\partial}{\partial x_7}\right)$$
$$H_1 = x_1\frac{\partial}{\partial x_1}-x_2\frac{\partial}{\partial x_2}-x_3\frac{\partial}{\partial x_3}+x_4\frac{\partial}{\partial x_4},\quad H_2 = x_3\frac{\partial}{\partial x_3}-x_4\frac{\partial}{\partial x_4}-x_5\frac{\partial}{\partial x_5}+x_6\frac{\partial}{\partial x_6},$$
$$H_3 = x_5\frac{\partial}{\partial x_5}-x_6\frac{\partial}{\partial x_6}-x_7\frac{\partial}{\partial x_7}+x_8\frac{\partial}{\partial x_8},\quad H_4 = x_5\frac{\partial}{\partial x_5}-x_6\frac{\partial}{\partial x_6}+x_7\frac{\partial}{\partial x_7}-x_8\frac{\partial}{\partial x_8},$$

acting on degree-r polynomials $p_r(x) := \prod_{i=1}^8 x_i^{\alpha_i}\in\mathbb{C}[x_1,\dots,x_8] = \bigoplus_{r=0}^\infty V_r$, where $\sum_{i=1}^8\alpha_i = r$. A suitable basis for V_r is given by the polynomials

$$p_r(x) = x_1^{a_1}x_2^{r-a_2}x_3^{a_3-a_1}x_4^{a_2-a_4}x_5^{a_5-a_3}x_6^{a_4-a_6-a_7}x_7^{a_7-a_5}x_8^{a_6}$$

subject to

$$0\le a_1,a_2\le r,\quad a_1\le a_3\le a_3+r,\quad a_2-r\le a_4\le a_2,$$
$$a_3\le a_5\le a_3+r,\quad a_5\le a_7\le a_5+r,\quad 0\le a_6\le a_4-a_7.$$

The generators may then be expressed as *raising* and *lowering* operators:

$$X_1|r;a\rangle = i((r-a_2)|r;a_2+1\rangle-(a_3-a_1)|r;a_1+1\rangle)$$
$$X_2|r;a\rangle = i((a_2-a_4)|r;a_4+1\rangle-(a_5-a_3)|r;a_3+1\rangle)$$
$$X_3|r;a\rangle = i((a_4-a_6-a_7)|r;a_6+1\rangle-(a_7-a_5)|r;a_5+1\rangle)$$
$$X_4|r;a\rangle = i((a_4-a_6-a_7)|r;a_7+1\rangle-a_6|r;a_5+1,a_6-1,a_7+1\rangle)$$

$$Y_1|r;a\rangle = i(a_1|r;a_1-1\rangle-(a_2-a_4)|r;a_2-1\rangle)$$
$$Y_2|r;a\rangle = i((a_3-a_1)|r;a_3-1\rangle-(a_4-a_6-a_7)|r;a_4-1\rangle)$$
$$Y_3|r;a\rangle = i((a_5-a_3)|r;a_5-1\rangle-a_6|r;a_6-1\rangle)$$
$$Y_4|r;a\rangle = i((a_5-a_3)|r;a_5-1,a_6+1,a_7-1\rangle-(a_7-a_5)|r;a_7-1\rangle)$$

Further, the resulting eigenvalues recover the symmetry in the $\mathfrak{so}_8$ Cartan matrix:

$$\begin{array}{lllll} H_1: & +2(a_1+a_2) & -1(a_3+a_4) & & -r \\ H_2: & -1(a_1+a_2) & +2(a_3+a_4) & -1(a_5+a_6) & -1(a_7) \\ H_3: & & -1(a_3+a_4) & +2(a_5+a_6) & \\ H_4: & & -1(a_3+a_4) & & +2(a_7) \end{array}$$
◂

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E. Miranda is supported by the Catalan Institution for Research and Advanced Studies via by the ICREA Academia Prizes 2016 and 2021. E. Miranda is also supported by the Spanish State Research Agency, through the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (project CEX2020-001084-M). A. Cavallar is supported by a collaboration grant of the Spanish Ministry of Education and by a Cellex Foundation grant.