Existence and classification of b-contact structures

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Introduction

A Jacobi structure on a manifold M is given by a pair (Λ, E) of a bivector field and vector field satisfying

$$[\Lambda,\Lambda]=2E\wedge\Lambda,\quad [\Lambda,E]=0.$$

They generalize contact and conformally symplectic structures. We study in three dimensions the class of Jacobi structures with the simplest singularity.

Definition 1. A b-Jacobi structure on a manifold M of dimension $2n+1$ is a Jacobi structure such that $E\wedge\Lambda^n \pitchfork 0$.

This implies that the hypersurface $Z=(E\wedge\Lambda^n)^{-1}(0)$ is foliated by codimension one conformally symplectic leaves and codimension two contact leaves, and $M\setminus Z$ are contact leaves of maximal dimension.

1 A formulation in terms of b-geometry

Inspired by the symplectic formulation of b-Poisson structures [GMP], in [MO] it was established that b-Jacobi structures admit a formulation in terms of contact forms on the b-tangent bundle. Given an orientable hypersurface Z on a manifold M, a b-vector field is a vector field on M tangent to Z.

Definition 2. The space of b-vector fields corresponds to the space of sections of a vector bundle on M called the b-tangent bundle and denoted by ${}^bTM$.

The dual of the b-tangent bundle denoted ${}^bT^*M$ is called *b-cotangent bundle*. Sections of $\bigwedge^k {}^bT^*M$ are called b-forms, and can be endowed with an exterior derivative operator.

Definition 3. A b-contact form in the b-manifold (M,Z) is a one b-form $\alpha\in {}^bT^*M$ satisfying $\alpha\wedge(d\alpha)^n\neq 0$, as a section of $\bigwedge^{2n+1}{}^bT^*M$.

The kernel of α defines b-contact structure, which is a codimension one subbundle of ${}^bTM$. The duality between b-contact forms and b-Jacobi structures was proved in [MO], and is reminiscent of the duality between b-symplectic forms and b-Poisson structures.

Example 4. Consider in $\mathbb{R}^3$, with $Z=\{z=0\}$ the b-contact form, and associated b-contact structure

$$\alpha=\frac{dz}{z}+xdy,\qquad \xi=\langle\frac{\partial}{\partial x}, x.z\frac{\partial}{\partial z}-\frac{\partial}{\partial y}\rangle$$

and the associated b-Jacobi structure is

$$E=z\frac{\partial}{\partial z},\qquad \Pi=\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}$$

The in $z=0$ belong to a conformally symplectic leaf.

Example 5. Consider in $\mathbb{R}^3$, with $Z=\{z=0\}$ the b-contact form and associated b-contact structure

$$\alpha=dx+y\frac{dz}{z},\qquad \xi=\langle y\frac{\partial}{\partial x}-z\frac{\partial}{\partial z},\frac{\partial}{\partial y}\rangle$$

and the associated b-Jacobi structure is

$$E=\frac{\partial}{\partial x},\qquad \Pi=z\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}+y\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}$$

The points in $\{y=0,z=0\}$ belong to a contact leaf, the points in $\{y\neq 0, z=0\}$ belong to conformally symplectic leaves.

In three dimensions, one can think of the induced structure on the critical surface Z as a b-Poisson structure (i.e. a Poisson vector field that vanishes transversely along a collection of closed curves).

2 Convexity in b-contact geometry

In three-dimensional contact topology, a very important role is played by the so-called "convex" surfaces. An embedded surface Σ is convex if there is a neighborhood $\Sigma\times(-\varepsilon,\varepsilon)$ where the contact structure is defined by a $\mathbb{R}$-invariant contact form. The same can be defined for a hypersurface on a high-dimensional contact manifold. In analogy, we can define

Definition 6. Let ξ be a b-contact structure in (M,Z). We say that it is convex along Z if it Z admits a neighborhood where ξ is defined by a $\mathbb{R}$-invariant b-contact form.

This means that ξ is given in $Z\times(-\varepsilon,\varepsilon)$ by the kernel of a one b-form that writes

$$\alpha=u\frac{dz}{z}+\beta,$$

with $u,\beta\in\mathbf{C}^\infty(\mathbf{Z})$. An important observation, unique to b-contact geometry, is that b-contact structures are always convex along Z.

Proposition 7. *Let ξ be a b-contact structure on an odd-dimensional b-manifold (M,Z). Then ξ is convex along Z.*

This fact is not true for an arbitrary hypersurface on a contact manifold, and follows in this setting by the difference between the b-contact and the contact condition on Z.

3 Existence theorems

We address now the existence of b-Jacobi structures on a prescribed three-dimensional b-manifold (M,Z). It was previously shown [MO] that given any connected surface Z, we denote by $\tilde Z$ two parallel disjoint copies of Z and then $(M,\tilde Z)$ admits some b-contact structure. We show instead an existence h-principle, where the hypersurface, the induced singular foliation and homotopy class of the b-plane field can be prescribed.

Theorem 1. *Let (M,Z) be a b-manifold of dimension 3 and let η be a b-plane field. Then η is homotopic through b-plane fields to a b-contact structure whose set of one-dimensional leaves is any collection of curves in Z such that*

$$\chi(Z_+)-\chi(Z_-)=\langle e_Z(\eta),Z\rangle. \tag{8}$$

Here Z_+,Z_- are the (possibly disconnected) surfaces with boundary (and induced sign by the coorientation of Γ) obtained from cutting Z open alongΓ, and χ denotes the Euler characteristic. We point out that if a b-contact structure ξ induces on Z a set of one-dimensional leaves as above, then equation (8) holds. Hence the theorem above shows that any singular foliation can arise if the three-dimensional leaves are **overtwisted**. This means that there is an embedded disk whose boundary is tangent to ξ, an object that plays an important role in the flexible aspects of contact topology.

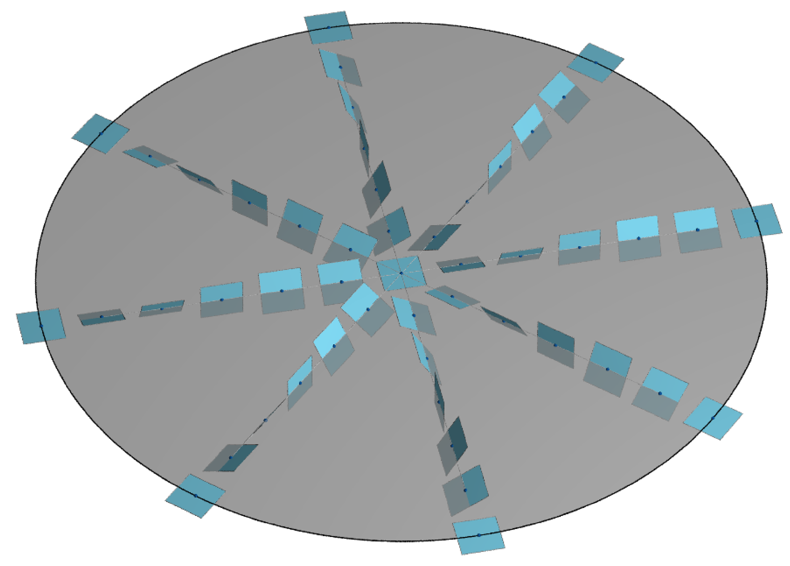

Figure 1. An overtwisted disk, by Patrick Massot

The study of non-overtwisted contact manifolds (also known as "tight") is much more subtle and rigid, some three-manifolds do not even admit a tight contact structure (Etnyre-Honda). We can show the following weaker existence result.

Theorem 2. *Let (M,Z) be a closed oriented three-dimensional b-manifold. There exists a positive b-contact structure that is tight in each connected component.*

One might transform any the tight leaf to an overtwisted one at will by performing a "'Lutz twist", without changing the homotopy class of the b-plane field.

Example 9. A tight b-contact structure in (S^3,S^2) can be constructed as follows. In $S^2\times(-\varepsilon,\varepsilon)$ we consider the $\mathbb{R}$-invariant b-contact form obtained by singularizing the standard tight convex sphere.

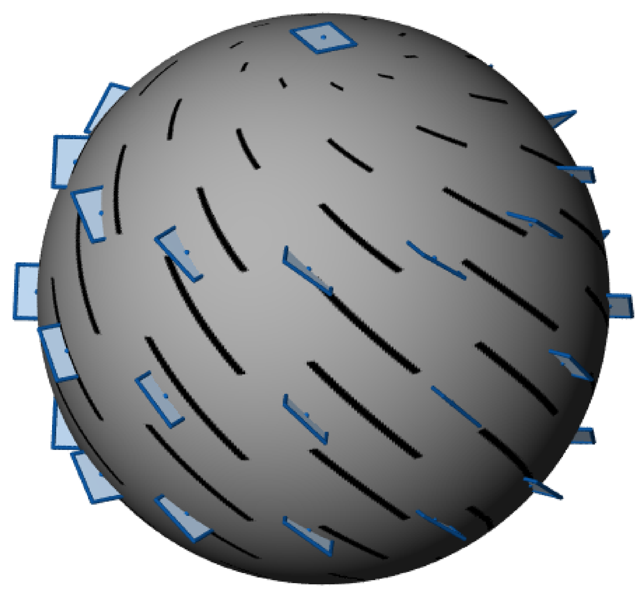

Figure 2. The standard tight convex sphere, by Patrick Massot

Now we fill each hemisphere with a disk equipped with the standard contact structure $\xi_{std}=\ker dz+ydx$ or the negative standard contact structure $\ker dz-ydx$.

4 Classification of b-contact structures

Our previous discussion gives a recipe to classify b-contact structures on a given b-manifold (M,Z). These results can be compared with the seminal work of Radko, where b-Poisson structures on closed surfaces were classified.

Recipe to classify b-contact structures:

1. the isotopy class of the dividing set (one-dimensional leaves) determines the structure near Z. First, we study which isotopy classes of one-dimensional leaves can arise on a given homotopy class of b-plane fields. This can be done using convex surface theory in contact topology
2. The tight/ovetwisted dichotomy of the three-dimensional leaves is influenced by the isotopy class of one-dimensional leaves. This follows from convexity and Giroux's theory.
3. Each connected component of $M\setminus \mathrm{Op}(Z)$ can be understood as a contact manifold with boundary.
4. If a component is overtwisted, it is determined by the relative homotopy class of the plane field.
5. If a component is tight, we need a classification of tight contact structures on that manifold with convex boundary.

We use this recipe to classify b-contact structures in some prototypical examples of b-manifolds like S^3 with $Z\cong S^2$ an equator, or with $Z\cong\mathbb{T}^2$ an unknotted torus on the three-sphere.

Theorem 3. *The classification of positive b-contact structures in (S^3,S^2), endowed with some orientation, up to b-isotopy is the following.*

- *There exists only one tight b-contact structure on (S^3,S^2). On the critical surface there is a single one-dimensional leaf.*
- *b-contact structures that are overtwisted in one connected component of $S^3\setminus S^2$ but tight in the other one are determined by a sign and an element in $\mathbb{Z}$. Again, on the critical surface there is a single one-dimensional leaf.*
- *For each Γ isotopy class of cooriented separating closed curves in S^2 satisfying $\chi(S^2_+)=\chi(S^2_-)$, fully overtwisted b-contact structures inducing Γ as set of one-dimensional leaves are parametrized by $\mathbb{Z}\oplus\mathbb{Z}$.*

Isotopy classes of cooriented curves that can arise as the set of one-dimensional leaves of a b-contact structure can be characterized combinatorically: they are in correspondence with two-equicolored trees with an even number of vertices.

References

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