KU LEUVEN

# Multisymplectic observables and higher Courant algebroids

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1. The higher case

# Multisymplectic forms

Let $M$ be a manifold, $n \geq 1$.

**Definition**

An $n$-plectic form is a closed, non-degenerate $\omega \in \Omega^{n+1}(M)$.

Examples:

i) Volume forms
ii) $(\wedge^n T^* B, d\theta_{can})$

Next: $\omega$ induces

A) the observables $L_\infty(M, \omega)$
B) a higher Courant algebroid $E \rightsquigarrow L_\infty(E)_\omega$

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B) a higher Courant algebroid $E \rightsquigarrow L_\infty(E)_\omega$

# A) Observables

**Definition** (Rogers 2012)

The $L_\infty$-algebra $L_\infty(M,\omega)$ of observables is

$$\underset{-n+1}{C^\infty(M)} \xrightarrow{d} \dots \xrightarrow{d} \Omega^{n-2}(M) \xrightarrow{d} \underset{0}{\Omega^{n-1}_{ham}(M,\omega)}$$

with higher brackets

$$\{\alpha_1,\dots,\alpha_k\}_k = \pm\iota_{X_{\alpha_k}} \dots \iota_{X_{\alpha_1}}\omega$$

# B) Higher Courant algebroids

Consider the "higher Courant algebroid" twisted by $\omega$:

$$(E, \langle\cdot,\cdot\rangle, pr_{TM}, [\![\cdot,\cdot]\!])$$

with

$$E := TM \oplus \wedge^n T^* M$$

$$\langle\cdot,\cdot\rangle \colon E \times E \to \wedge^{n-1} T^* M$$

$$[\![X + \alpha, Y + \beta]\!] = [X, Y] + \mathcal{L}_X \beta - \mathcal{L}_Y \alpha - \frac{1}{2} d(\iota_X \beta - \iota_Y \alpha) + \iota_Y \iota_X \omega.$$

**Definition** (Z. 2012 after Fiorenza-Manetti 2007, Getzler 2010)

The $L_\infty$-algebra $L_\infty(E)_\omega$ is

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with higher brackets:

- binary bracket:

  for $e_i \in \Gamma(E)$ the twisted Courant bracket

$$[e_1, e_2] = [\![e_1, e_2]\!],$$

  for $e = (X, \alpha) \in \Gamma(E)$ and $\xi \in \Omega^{\bullet \leq n-2}(M)$

$$[e, \xi] = \frac{1}{2}\mathcal{L}_X \xi.$$

- trinary bracket:

  for $e_i \in \Gamma(E)$

$$[e_0, e_1, e_2] = -\frac{1}{6}\left(\langle [\![e_0, e_1]\!], e_2 \rangle + c.p.\right).$$

  for $\xi \in \Omega^{\bullet \leq n-2}(M)$ and $e_i = (X_i, \alpha_i) \in \Gamma(E)$

$$[\xi, e_1, e_2] = -\frac{1}{6}\left(\frac{1}{2}(\iota_{X_1}\mathcal{L}_{X_2} - \iota_{X_2}\mathcal{L}_{X_1}) + \iota_{[X_1, X_2]}\right)\xi.$$

- $n$-ary bracket for $n \geq 3$ with $n$ an odd integer:

  for $e_i = (X_i, \alpha_i) \in \Gamma(E)$, $[e_0, \cdots, e_{n-1}] = \sum_i [X_0, \ldots, \alpha_i, \ldots, X_{n-1}]$, with

$$[\alpha, X_1, \ldots, X_{n-1}] = b_{n-1} \sum_{i<j} (-1)^{i+j+1} \iota_{X_{n-1}} \ldots \widehat{\ } \ldots \iota_{X_1} [\alpha, X_i, X_j],$$

  for $\xi \in \Omega^{\bullet \leq n-2}(M)$ and $e_i = (X_i, \alpha_i) \in \Gamma(E)$,

$$[\xi, e_1, \cdots, e_{n-1}] = b_{n-1} \sum_{i<j} (-1)^{i+j+1} \iota_{X_{n-1}} \ldots \widehat{\ } \ldots \iota_{X_1} [\xi, X_i, X_j].$$

Above $b_{n-1} := \frac{(-1)^{\frac{n+1}{2}} 12 B_{n-1}}{(n-1)(n-2)}$, where the $B_k$ are the Bernoulli numbers:

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

# Main theorem: an embedding of $L_\infty$-algebras

**Theorem (Miti-Z. 2022)**

*There is an $L_\infty$-embedding*

$$\Psi : L_\infty(M,\omega) \hookrightarrow L_\infty(E)_\omega$$

*with components*

$$\Psi_1(\xi \oplus \alpha) = \xi \oplus (X_\alpha, \alpha)$$

$$\Psi_k(\xi_1 \oplus \alpha_1, \dots, \xi_k \oplus \alpha_k) = B_{k-1} \sum_{j=1}^{k} (-1)^{k-j} \iota_{X_{\alpha_1}} \dots \widehat{\iota_{X_{\alpha_j}}} \dots \iota_{X_{\alpha_k}} (\xi_j \oplus \alpha_j)$$

*for $k \geq 2$. Here $\xi_i \oplus \alpha_i \in \Omega^{\leq n-2}(M) \oplus \Omega^{n-1}_{ham}(M,\omega)$.*

Remark: This was proven

- $n = 2$: by [Rogers 2013]
- existence for all $n$: unpublished preprint by [Ritter-Sämann 2015]

Remark: $\Psi$ is compatible with gauge transformations by invariant $B \in \Omega^n(M)$.

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# Sketch of proof

1) Vector space isomorphism

$$L_\infty(M,\omega) \overset{\cong}{\to} \underbrace{\Omega^{\leq n-2}(M) \oplus \{(X_\alpha, \alpha) : \alpha \in \Omega^{n-1}_{ham}(M,\omega)\}}_{=:\mathcal{A}} \subset L_\infty(E)_\omega$$

Get two $L_\infty[1]$-algebra structures $(\mathcal{A}[1], m_k)$ and $(\mathcal{A}[1], e_k)$.

2) Strategy:

A degree $0$ linear map $p\colon\ S^{\geq 1}\mathcal{A}[1] \to \mathcal{A}[1]$

$\rightsquigarrow$ degree $0$ coderivation $C_p$ of the coalgebra $S^{\geq 1}\mathcal{A}[1]$

$\rightsquigarrow$ automorphism

$$e^{C_p}\colon\ S^{\geq 1}\mathcal{A}[1] \to S^{\geq 1}\mathcal{A}[1].$$

Then:

- $D' := e^{C_p} \circ D_m \circ e^{-C_p}$ is also a coderivation on $S^{\geq 1}\mathcal{A}[1]$
- $e^{C_p}$ intertwines $D_m$ and $D'$

Get

- a new $L_\infty[1]$-algebra structure $m'$ on $\mathcal{A}[1]$ (the "push-forward")
- an $L_\infty[1]$-isomorphism from $(\mathcal{A}[1], m_k)$ to $(\mathcal{A}[1], m'_k)$.

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# Sketch of proof (cont.)

3) Ansatz:

As $p$ take the degree $0$ linear map

$$\sum_{j=0}^{\infty} c_j \langle\ ,\ \rangle_-^j \colon\ S^{\geq 1}\mathcal{A}[1] \to \mathcal{A}[1]$$

where $\langle\ ,\ \rangle_-$ corresponds to the skew-symmetric pairing on $E$.
We find coefficients $c_j$ so that $m'_k = e_k$.

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# Thank you for your attention