Non-principal T-duality

Aldo Witte, KU Leuven, 13th of June 2022, Madrid

T-duality

Joint work with Gil Cavalcanti.

Definition

Let $M, \widetilde{M}$ be principal T^k-bundles with base B, and $H, \widetilde{H}$ T^k-invariant closed three forms. They are **T-dual** if there exists a T^{2k}-invariant form $F \in \Omega^2(M \times_B \widetilde{M})$ such that

$$dF = p^* H - \widetilde{p}^* \widetilde{H}$$

and

$$F : \mathfrak{t}_M \times \mathfrak{t}_{\widetilde{M}} \to \mathbb{R},$$

with $\mathfrak{t}_M$ the tangent to the fibre of p, is non-degenerate.

Consequences

Theorem (Bouwknegt–Evslin–Mathai)

If (M, H) and $(\widetilde{M}, \widetilde{H})$ are T-dual via F, then

$$\tau : (\Omega^{\bullet}_{T^k}(M), d_H) \to (\Omega^{\bullet}_{T^k}(\widetilde{M}), d_{\widetilde{H}}), \quad \rho \mapsto \int_{T^k} e^F \wedge \rho,$$

with integral over fibres of p, is an isomorphism of chain complexes.

Theorem (Cavalcanti-Gualteri)

There is an isomorphism of Courant algebroids

$$(TM \oplus T^*M)/T^k \to (T\widetilde{M} \oplus T^*\widetilde{M})/T^k,$$

intertwining the Clifford actions on $\Omega^{\bullet}_{T^k}(M)$ and $\Omega^{\bullet}_{T^k}(\widetilde{M})$.

Consequences 2

Theorem

If (M, H) and $(\widetilde{M}, \widetilde{H})$ are T-dual, then τ sends Dirac structures to Dirac structures, generalized complex structures to generalized complex structures.

Example

$(S^3, 0)$ is T-dual to $(S^2 \times S^1, \sigma \wedge \theta)$, with σ the curvature of the Hopf fibration.

Toric actions

Definition

A **toric action** is an effective T^n-action on M^{2n} with connected isotropy groups.

Proposition (Cavalcani-Klaasse-W)

$\mu : M \to B := M/T^n$ is a manifold with corners, and if $I_{\partial B}$ are the functions vanishing at ∂B, then $I_{|D|} := \mu^* I_{\partial B}$ is an elliptic divisor.

Proposition

The infinitesimal generators $X_1, \ldots, X_n$ all lift to nowhere vanishing sections of $\mathcal{A}_{|D|}$.

Toric T-duality

Definition

Let $M^{2n}, \widetilde{M}^{2n}$ be toric actions with base B, and $H \in \Omega^3_{T^n}(M, \mathcal{A}_{|D|})$ closed, and similar for $\widetilde{H}$. They are **T-dual** if there exists a T^{2n}-invariant form $F \in \Omega^2(M \times \widetilde{M}, \mathcal{A}_{|D|} \times \mathcal{A}_{|\widetilde{D}|})$ such that

$$\iota^*_{(M\setminus D)\times_B(\widetilde{M}\setminus\widetilde{D})}(dF - q^*H + \widetilde{q}^*H) = 0$$

and

$$F : \mathfrak{t}_M \times \mathfrak{t}_{\widetilde{M}} \to \mathbb{R},$$

with $\mathfrak{t}$ the tangent to the fibre of p, is non-degenerate.

Consequences

Theorem

If (M, H) and $(\widetilde{M}, \widetilde{H})$ are T-dual via F, then

$$\tau : (\Omega^{\bullet}_{T^k}(M, \mathcal{A}_{|D|}), d_H) \to (\Omega^{\bullet}_{T^k}(\widetilde{M}, \mathcal{A}_{|\widetilde{D}|}), d_{\widetilde{H}}),$$
$$\rho \mapsto \iota_{X_1 \wedge \cdots \wedge X_k}(e^F \wedge \rho),$$

with $X_1, \ldots, X_k$ the infinitesimal generators of the action on M, is an isomorphism of chain complexes.

Consequences 2

Theorem

There is an isomorphism of Courant algebroids

$$(\mathcal{A}_{|D|} \oplus \mathcal{A}^*_{|D|})/T^k \to (\mathcal{A}_{|\widetilde{D}|} \oplus \mathcal{A}^*_{|\widetilde{D}|})/T^k,$$

intertwining the actions on $\Omega^\bullet_{T^k}(M, \mathcal{A}_{|D|})$ *and* $\Omega^\bullet_{T^k}(\widetilde{M}, \mathcal{A}_{|\widetilde{D}|})$.

Generalized complex structures on $\mathcal{A}_{|D|}$ are send to generalized complex structures on $\mathcal{A}_{|\widetilde{D}|}$. But these might not descend to M.

Weak classification of torus actions

Definition

Two toric actions $M, \widetilde{M}$ over the same base B are **locally equivalent**, if each $b \in B$ has a neighbourhood V such that $p^{-1}(V)$ and $\widetilde{p}^{-1}(V)$ are equivariantly diffeomorphic.

Note, that one works with a fixed T^n acting.

Theorem (Haefliger-Salem)

Given a toric action M with base B, locally equivalent actions $\widetilde{M}$ correspond to classes in $H^2(B; \mathbb{Z}^n)$.

Existence of T-duals

Let $\theta \in \Omega^1(\mathcal{A}_{|D|}; \mathfrak{t})$ be a connection-one form for M. Then

$$H = \langle q^*\widetilde{c}, \theta\rangle + q^*b, \quad \widetilde{c} \in \Omega^2(\log \partial B; \mathfrak{t}^*), b \in \Omega^3(\log \partial B).$$

Want to use $[\widetilde{c}]$ to build a new toric action.

Theorem (Gualteri-Li-Pelayo-Ratiu)

$H^2(\log \partial B) \simeq H^2(B) \oplus H^1(\partial B[1]) \oplus H^0(\partial B[2]),$

Therefore:

Theorem

If M is a toric action and $[\widetilde{c}] \in H^2(B; \mathbb{Z}^n)$, then there exists a T-dual.

Proof

- $H = \langle q^* \widetilde{c}, \theta \rangle + q^* b$, and let $\widetilde{M}$ be the locally equivalent action corresponding to $[\widetilde{c}] \in H^2(B; \mathbb{Z}^n)$.
- Let $\widetilde{\theta} \in \Omega^1(\mathcal{A}_{|\widetilde{D}|}; \mathfrak{t}^*)$ be a connection one-form with $d\widetilde{\theta} = \widetilde{c}$.
- Let $\widetilde{H} = \langle \widetilde{q}^* d\theta, \widetilde{\theta} \rangle + q^* b$.
- Then $p^* H - \widetilde{p}^* \widetilde{H} = -d\langle \theta, \widetilde{\theta} \rangle$.
- Therefore $F := -\left\langle \theta, \widetilde{\theta} \right\rangle$ will provide the T-dual.

More T-duals

The previous construction misses a lot of T-duals.

Theorem

Let $(M, 0)$ and $(\widetilde{M}, 0)$ be toric actions, with the same contractible base B. Then they are T-dual.

New source of topology change: singularities of the action.

Proof idea

- Fix a point $b \in B$, and use the normal form of the actions on a neighbourhood.
- In this local normal form, write down an explicit F. Which can be taken to have constant coefficients.
- Because this form has constant coefficients, and the base is contractible it can be extended as a closed form to the entire manifold.

Outlook

- More examples.
- Other non-principal torus actions
- Stable under smoothing of divisor?
- Relate with HMS approaches for stable gc manifolds.

References

Peter Bouwknegt, Jarah Evslin, and Varghese Mathai.
T-duality: Topology change from h-flux.
Communications in Mathematical Physics, 08 2004.

Gil Cavalcanti and Marco Gualtieri.
Generalized complex geometry and t-duality.
CRM Proceedings and Lecture Notes, pages 341–366, 06 2010.

Gil R. Cavalcanti, Ralph L. Klaasse, and Aldo Witte.
Fibrations in semi-toric and generalized complex geometry.
Canidian journal of mathematics, 2020.

Gil R. Cavalcanti, Ralph L. Klaasse, and Aldo Witte.
Self-crossing stable generalized complex structures.
Journal of symplectic geometry, 2020.

Marco Gualtieri, Songhao Li, Álvaro Pelayo, and Tudor S. Ratiu.
The tropical momentum map: a classification of toric log symplectic manifolds.
Mathematische Annalen, pages 1217–1258, 04 2017.

André Haefliger and Eliane Salem.
Actions of tori on orbifolds.
Annals of Global Analysis and Geometry, 02 1991.