Higher Chern Correspondence

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joint work with Luis Álvarez Cónsul and Mario García Fernández

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Outline

1. Motivation
2. Gerbes
3. Lie 2-Groups
4. 2-Principal Bundles
5. Conclusions

Motivation

The Hull-Strominger System

- The **Hull-Strominger System** over a (possibly non-Kähler) Calabi-Yau manifold (X, Ω) with a hermitian vector bundle (E, h) is

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\begin{aligned}
F_h \wedge \omega^{n-1} &= 0, \qquad & F_h^{0,2} &= 0, \\
d(||\Omega||_\omega \omega^{n-1}) &= 0, \qquad & dd^c\omega + \langle F_h \wedge F_h \rangle &= 0,
\end{aligned}
$$

  for unknowns

  1. $\omega \in \Lambda^{1,1}(X, \mathbb{R})$ such that $g := \omega(J\cdot, \cdot) > 0$,
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  2. A_h a unitary connection on (E, h).
- It can be reinterpreted [1, García-Fernández, Rubio, Tipler, 2020] as a system for **integrable horizontal lifts** on a **string Courant algebroid**, revealing more about its symmetries and geometry (moment map interpretation).

Connections on 2-Principal Bundles

- Conjecturally, smooth Courant algebroids are 'Atiyah algebroids' for higher principal bundles [2, Severa, 1998], [3, Collier, 2011], [4, Sheng, Xu, Zhu, 2017], but this is still not a precise statement.

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- We introduce **holomorphic** *String*-principal bundles and connections on them through a simple model based on **gerbes** and relate them to Courant algebroids.
- We reformulate the Hull-Strominger system as a system for connections on a *String*-bundle via a **higher Chern correspondence**.

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- $t_{ijk} : U_{ijk} \to T$, $A_{ij} \in \Lambda^1(U_{ij}, \mathfrak{t})$, $B_i \in \Lambda^2(U_i, \mathfrak{t})$ such that

$$\begin{aligned} t_{ijk} t_{ijl}^{-1} t_{ikl} t_{jkl}^{-1} &= 1 && \text{on } U_{ijkl}, \\ A_{ij} - A_{ik} + A_{jk} &= t_{ijk}^* \theta_{MC} && \text{on } U_{ijk}, \\ B_i - B_j &= -dA_{ij} && \text{on } U_{ij}. \end{aligned}$$

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Its **curvature** is $H \in \Lambda^3(X, \mathfrak{t})$ defined locally as $H_{|U_i} := dB_i$.

It is **holomorphic with compatible curving** if t_{ijk} and A_{ij} are holomorphic and $B_i \in \Lambda^{2,0}(U_i, \mathfrak{t})$.

Isomorphisms of Gerbes

Definition

An **isomorphism of T-gerbes with con. str.** $\varphi : (\mathcal{L}', \nabla') \to (\mathcal{L}, \nabla)$ is described by

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$$A'_{ij} - A_{ij} = t_{ij}^* \theta_{MC} - A_i + A_j \qquad \text{on } U_{ij}.$$

Its **curvature** with respect to curvings $\{B'_i\}_{i \in I}$, $\{B_i\}_{i \in I}$ is $F_\varphi \in \Lambda^2(X, \mathfrak{t})$ defined locally as $F_{\varphi|U_i} := dA_i - B'_i + B_i$. (Note $dF_\varphi = H - H'$).

2-Isomorphisms of Gerbes

Definition

A **2-isomorphism of gerbes with con. str.**
$$\begin{tikzcd} (\mathcal{L}',\nabla') \arrow[r, bend left=50, "\varphi_1"] \arrow[r, bend right=50, "\varphi_2"'] \arrow[r, phantom, "\Downarrow"] & (\mathcal{L},\nabla) \end{tikzcd}$$
is described by

- $t_i : U_i \to T$ such that

$$t^{1}_{ij,1} t^{-1}_{ij,2} = t_i t_j^{-1} \qquad \text{on } U_{ij},$$
$$A_{i,1} - A_{i,2} = t_i^* \theta_{MC} \qquad \text{on } U_i.$$

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$\langle T^1$-gerbes with con. str. and curving$\rangle \to \langle T^2$-gerbes with con. str. and curving$\rangle$.
- A $U(1)$-**gerbe with con. str.** given by $(\{U_i\}, \{A_{ij}\}, \{t_{ijk}\})$ gives rise to an **exact Courant algebroid** by gluing $TU_i \oplus T^*U_i$ to $TU_j \oplus T^*U_j$ through the B-field dA_{ij}.

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  - Strict Lie 2-groups $=$ Lie Crossed Modules $(G, T, t, \triangleright)$.
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  - Strict Lie 2-groups = Lie Crossed Modules $(G, T, t, \triangleright)$.
  - **Central extensions** $BT \to \mathbb{G} \to G$.
- [5, Schommer-Pries, 2009] proved central extensions = multiplicative gerbes and showed that the *String* 2-group has a finite dimensional model of the form $BU(1) \to String(n) \to Spin(n)$.

Central Extensions of Lie 2-Groups

Definition [6, Carey et al., 2005], [5, Schommer-Pries, 2009]

A **central extension** $BT \to \mathbb{G} \to G$ is a T-gerbe over G, $\mathbb{G} \to G$, together with

- An isomorphism of T-gerbes over $G \times G$, $g_1^*\mathbb{G} \otimes g_2^*\mathbb{G} \xrightarrow{m} (g_1 g_2)^*\mathbb{G}$

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- A 2-isomorphism of T-gerbes over $G \times G \times G$,

$$\begin{tikzcd}[column sep=huge] g_1^*\mathbb{G} \otimes g_2^*\mathbb{G} \otimes g_3^*\mathbb{G} \arrow[r, bend left=60, "(g_1 g_2, g_3)^* m \circ (g_1, g_2)^* m"] \arrow[r, bend right=60, "(g_1, g_2 g_3)^* m \circ (g_2, g_3)^* m"'] \arrow[r, phantom, "\Downarrow \alpha"] & (g_1 g_2 g_3)^*\mathbb{G} \end{tikzcd}$$

such that $\alpha(g_1, g_2, g_3)\alpha(g_1, g_2 g_3, g_4)\alpha(g_2, g_3, g_4) = \alpha(g_1 g_2, g_3, g_4)\alpha(g_1, g_2, g_3 g_4)$.

Connective Central Extensions

The 2-groups that relate to generalized geometry are **enhancements** of central extensions $BT \to \mathbb{G} \to G$.

Definition [7, Waldorf, 2008], [A.C., G.F., T., 2022]

A **connective central extension** $BT_\nabla \to \mathbb{G}_\nabla \to G$ is a central extension $BT \to \mathbb{G} \to G$ as before, where $\mathbb{G}$ is equipped with a connective structure and m, α are an isomorphism and a 2-isomorphism of 'gerbes with connective structure'.

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It is **holomorphic** if G, T, the gerbe and its connective structure are holomorphic, and m, α preserve the holomorphic structure.

Connective Central Extensions

Let θ^L, θ^R be the left- and right-invariant Maurer-Cartan 1-forms on G.

Theorem [7, Waldorf, 2008], [A.C., G.F., T., 2022]

1. Any Ad-invariant symmetric bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{t}$ satisfying a certain integrality condition determines a connective central extension $BT_\nabla \to \mathbb{G}_\nabla \to G$.

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2. A connective central extension $BT_\nabla \to \mathbb{G}_\nabla \to G$ determines canonically an Ad-invariant symmetric bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{t}$ and these constructions are inverse to each other.

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3. A connective central extension $BT_\nabla \to \mathbb{G}_\nabla \to G$ has a canonical curving B^θ on $\mathbb{G}_\nabla$ such that
   - The curvature of $m : g_1^*\mathbb{G}_\nabla \otimes g_2^*\mathbb{G}_\nabla \to (g_1 g_2)^*\mathbb{G}_\nabla$ is $-\langle g_1^*\theta^L \wedge g_2^*\theta^R \rangle \in \Lambda^2(G \times G, \mathfrak{t})$.
   - The curvature of $\mathbb{G}_\nabla$ is $\frac{1}{6}\langle \theta^L \wedge [\theta^L \wedge \theta^L] \rangle \in \Lambda^3(G, \mathfrak{t})$.

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We call the pair (θ^L, B^θ) the **Maurer-Cartan 1-form** on $\mathbb{G}_\nabla$.

Holomorphic Connective Central Extensions

Proposition [A.C., G.F., T., 2022]

$BT_\nabla \to \mathbb{G}_\nabla \to G$ is holomorphic $\Leftrightarrow$ G, T are holomorphic and $\langle \cdot, \cdot \rangle$ is $\mathbb{C}$-linear. In this case, B^θ is holomorphic.

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Definition [A.C., G.F., T., 2022]

The **complexification** of a central extension of Lie 2-groups $BT_\nabla \to \mathbb{K}_\nabla \to K$ classified by $\langle \cdot, \cdot \rangle : \mathfrak{k} \otimes \mathfrak{k} \to \mathfrak{t}$ is the holomorphic central extension $B(T^{\mathbb{C}})_\nabla \to \mathbb{K}^{\mathbb{C}}_\nabla \to K^{\mathbb{C}}$ classified by the $\mathbb{C}$-linear extension of $\langle \cdot, \cdot \rangle$.

2-Principal Bundles

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Definition [8, Waldorf, 2009], [A.C., G.F., T., 2022]

Let $BT_\nabla \to \mathbb{G}_\nabla \to G$ be a central extension. A $\mathbb{G}_\nabla$-**principal bundle** over M is a $G \to P \to M$ together with a T-gerbe with con. str. $\mathbb{P}_\nabla \to P$ and

- An isomorphism over $P \times G$, $p^*\mathbb{P}_\nabla \otimes g^*\mathbb{G}_\nabla \overset{act}{\to} (pg)^*\mathbb{P}_\nabla$

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- An isomorphism over $P \times G$, $p^*\mathbb{P}_\nabla \otimes g^*\mathbb{G}_\nabla \xrightarrow{act} (pg)^*\mathbb{P}_\nabla$
- A 2-isomorphism over $P \times G \times G$,

$$\begin{tikzcd}[column sep=huge]
p^*\mathbb{P}_\nabla \otimes g_1^*\mathbb{G}_\nabla \otimes g_2^*\mathbb{G}_\nabla \arrow[r, bend left=50, "(pg_1{,}g_2)^*act \circ (p{,}g_1)^*act"{name=U}] \arrow[r, bend right=50, "(p{,}g_1g_2)^*act \circ (g_1{,}g_2)^*m"'{name=D}] & (pg_1g_2)^*\mathbb{P}_\nabla
\arrow[Rightarrow, from=U, to=D, "\rho"]
\end{tikzcd}$$

such that $\rho(p, g_1, g_2)\rho(p, g_1g_2, g_3)\alpha(g_1, g_2, g_3) = \rho(pg_1, g_2, g_3)\rho(p, g_1, g_2g_3)$.

Connections on 2-Principal Bundles

Definition [8, Waldorf, 2009], [A.C., G.F., T., 2022]

A **connection** on $(P, \mathbb{P}_\nabla, act, \rho)$ is a pair (A, B), where $A \in \Omega^1(P, \mathfrak{g})$ is a connection on P and B is a curving on $\mathbb{P}_\nabla \to P$ such that the curvature of $act : p^*\mathbb{P}_\nabla \otimes g^*\mathbb{G}_\nabla \to (pg)^*\mathbb{P}_\nabla$ with respect to $(p^*B \otimes g^*B^\theta, (pg)^*B)$ is $-\langle p^*A \wedge g^*\theta^R \rangle \in \Lambda^2(P \times G, \mathfrak{t})$.

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Proposition: Bianchi Identity

If (A, B) is a connection, then the curvature of $\mathbb{P}_\nabla \to P$ with respect to B is $\hat{H} = -\pi^*H - CS_3(A) \in \Lambda^3(P, \mathfrak{t})$ for some $H \in \Lambda^3(M, \mathfrak{t})$ with $dH + \langle F_A \wedge F_A \rangle = 0$, where $CS_3(A) := \langle dA \wedge A \rangle + \frac{1}{3}\langle A \wedge [A \wedge A] \rangle$.

The pair $(F_A, H) \in \Lambda^2(M, ad\, P) \oplus \Lambda^3(M, \mathfrak{t})$ is the **curvature** of (A, B).

2-Principal Bundles and Courant Algebroids

Proposition [A.C., G.F., T., 2022]

A $\mathbb{G}_\nabla$-principal bundle $(P, \mathbb{P}_\nabla, act, \rho)$ gives rise to a Courant algebroid E by a reduction procedure as in [9, Bursztyn, Cavalcanti, Gualtieri, 2005], [10, García-Fernández, 2013].

A connection on $(P, \mathbb{P}_\nabla, act, \rho)$ gives a splitting $E = T^*M \oplus ad\, P \oplus TM$.

Connections on Holomorphic 2-Principal Bundles

Let $BT_\nabla \to \mathbb{G}_\nabla \to G$ be a holomorphic extension.

Definition [A.C., G.F., T., 2022]

Let $(P, \mathbb{P}_\nabla, act, \rho)$ be a holomorphic $\mathbb{G}_\nabla$-bundle and (A, B) a connection on it, we say that it is **compatible with the holomorphic structure** if $A^{0,1} = \bar{\partial}_P$ and B can be represented by $(2,0)$ forms on P.

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Proposition [A.C., G.F., T., 2022]

1. The curvature (F_A, H) of a connection that is compatible with a holomorphic structure satisfies $F_A^{0,2} = 0$, $H^{0,3+1,2} = 0$.

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1. The curvature (F_A, H) of a connection that is compatible with a holomorphic structure satisfies $F_A^{0,2} = 0$, $H^{0,3+1,2} = 0$.
2. Conversely, if $(P, \mathbb{P}_\nabla, act, \rho)$ is a smooth $\mathbb{G}_\nabla$-bundle and (A, B) is a connection on it such that $F_A^{0,2} = 0$, $H^{0,3+1,2} = 0$, then there exists a holomorphic $\mathbb{G}_\nabla$-bundle with compatible connection isomorphic to $(P, \mathbb{P}_\nabla, act, \rho, A, B)$ as smooth bundles with connection. It is unique up to holomorphic isomorphism preserving connections.

The Chern Correspondence

Let $BU(1)_\nabla \to \mathbb{K}_\nabla \to K$ be a central extension with K compact, let $(P_h, \mathbb{P}_{\nabla,h}, act, \rho)$ be a $\mathbb{K}_\nabla$-bundle over a complex manifold M and let $\mathbb{P}^{\mathbb{C}}_{\nabla,h}$ be its complexification.

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Theorem [A.C., G.F., T., 2022]

There is a one-to-one correspondence between:

- $((A_h, B_h), \omega)$, where $\omega \in \Lambda^{1,1}(M, \mathbb{R})$ and (A_h, B_h) is a connection on $\mathbb{P}_{\nabla,h}$ with $F^{0,2}_{A_h} = 0$, $H_h = d^c\omega$.
- Holomorphic structures on $\mathbb{P}^{\mathbb{C}}_{\nabla,h}$.

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Moreover, the holomorphic bundle corresponding to $((A_h, B_h), \omega)$ carries a canonical compatible connection with curvature $(F_{A_h}, -2i\partial\omega)$.

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- Holomorphic structures on $\mathbb{P}^{\mathbb{C}}_{\nabla,h}$.

Moreover, the holomorphic bundle corresponding to $((A_h, B_h), \omega)$ carries a canonical compatible connection with curvature $(F_{A_h}, -2i\partial\omega)$.

We define an **integrable connection** on $(P_h, \mathbb{P}_{\nabla,h}, act, \rho)$ to be a pair $((A_h, B_h), \omega)$ such that $F^{0,2}_{A_h} = 0$ and $H_h = d^c\omega$ (Note Bianchi $\Rightarrow dd^c\omega + \langle F_h \wedge F_h\rangle = 0$).

Conclusions

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- **Question**: Can we interpret solutions to these equations as zeroes of some moment map on the space of integrable connections? Can we perform a (higher?) symplectic reduction giving a moduli space of solutions?

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Thanks!