G2 structure manifolds and almost contact structures

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Generalized Geometry in Interaction, Madrid

Based on work with Xenia de la Ossa and Matthew Magill (2101.12605)

Motivation and summary

This talk:
Explore the connection between G_2 structures and almost contact structures

Key points:

- Almost contact structures are always present and reduce the structure group.
- Rarely used explicitly in the construction of G_2 (structure) manifolds.
- Effect on string constructions involving G_2 (structure) manifolds?

Motivation and summary

Motivation

Use string theory to probe geometry, deformations, invariants,...

Strings on 7-dim manifolds with G_2 structure

- 4D $\mathcal{N} = 1$ vacua from M theory
  - G_2 holonomy: Minkowski (singularities $\rightsquigarrow$ particle physics)
  - G_2 structure: AdS
- 3D $\mathcal{N} = 1$ heterotic string vacua (Minkowski/AdS):
  - integrable G_2 structure, $V \to Y$ G_2 instanton bundle.
- 3D $\mathcal{N} = 1$ type II vacua (Minkowski/AdS):
  - G_2 structure; branes and orientifolds as needed.

Outline

Setting the stage

Almost contact structures

Sasaki:60,61

Let Y be odd-dimensional, Riemannian manifold, with metric g.
Y has an *almost contact structure* (J, R, σ) if it admits

- endomorphism J of the tangent bundle TY
- unit vector field R (with respect to the metric g),
- a 1-form σ (the *contact form*)

satisfying

$$J^2 = -\mathbf{1} + R \otimes \sigma \ , \quad \sigma(R) = 1 \ .$$

The ACS is metric (ACMS) if furthermore

$$g(Ju, Jv) = g(u, v) - \sigma(u)\,\sigma(v) \ , \qquad \forall u, v \in \Gamma(TY) \ .$$

Almost contact structures

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Comment:
Odd-dimensional manifolds always admit a nowhere vanishing vector field ($\chi = 0$)

Hopf:27

Moreover, the ACS is a **contact structure** if *...,Sparks:10*

$$\sigma \wedge \mathrm{d}\sigma \wedge \cdots \mathrm{d}\sigma \neq 0 \ .$$

G_2 structures

Bonan:66, Fernandez–Gray:82, Bryant:87,03, Fernandez–Ugarte:98, Hitchin:00, Joyce:00, Friedrich–Ivanov:01, Gauntlett et.al.:01, Chiossi–Salamon:02, ...

- (Y, φ) has G_2 *structure* specified by non-degenerate positive 3-form φ

  True whenever Y is orientable and spin $\rightsquigarrow$ Y admits a nowhere-vanishing spinor η.

- Locally

$$\varphi_0 = (e^{12} + e^{34} + e^{56}) \wedge e^7 + e^{135} - e^{146} - e^{236} - e^{245} \, .$$

- $\varphi \to$ Riemannian metric g_φ on Y, and a 4-form $\psi = *\varphi$

- Torsion classes $\sim$ irreps of G_2:

$$\begin{aligned} \mathrm{d}\varphi &= \tau_0 \, \psi + 3 \, \tau_1 \wedge \varphi + *\tau_3 \, , \qquad & \Lambda^4 &= \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27} \, , \\ \mathrm{d}\psi &= 4 \, \tau_1 \wedge \psi + *\tau_2 \, , \qquad & \Lambda^5 &= \Lambda^5_7 \oplus \Lambda^5_{14} \, . \end{aligned}$$

- G_2 holonomy $\iff \mathrm{d}\varphi = 0 = \mathrm{d}\psi \iff \nabla_{LC}\eta = 0$

G_2, ACMS and $SU(3)$ structures

G_2, ACMS and $SU(3)$ structures

,...Arikan et.al:11,12, Todd:15

Any 7-dimensional G_2 structure manifold (Y, φ) admits (J, R, σ):

- a nowhere vanishing vector field R
- a nowhere vanishing one-form $\sigma = g_\varphi(R, \cdot)$
$$\text{w.l.o.g. } \sigma(R) = g_\varphi(R, R) = 1$$
- an endomorphism J on TY: $J(u) = R \times_\varphi u$, $\quad \forall\, u \in \Gamma(TY)$
$$J^a{}_b = -\varphi^a{}_{bc} R^c$$
- and furthermore a fundamental two form $\omega = i_R(\varphi)$

and the G_2 metric satisfies $g(Ju, Jv) = g(u, v) - \sigma(u)\,\sigma(v)$,$\forall u, v \in \Gamma(TY)$.

So, all G_2 structure manifold (Y, φ) admits an almost contact metric structure.

Proving ACS properties is straightforward. E.g. $J^2 = -\mathbf{1} + R \otimes \sigma$:

$$J^a{}_b J^b{}_c = \varphi^a{}_{bd} \varphi^b{}_{ce} R^d R^e = \ldots = -\delta^a_c + R^a \sigma_c$$

G_2, ACMS and $SU(3)$ structures

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So, all G_2 structure manifold (Y, φ) admits an almost contact metric structure.

Contact structure requires $\sigma \wedge \mathrm{d}\sigma \wedge \mathrm{d}\sigma \wedge \mathrm{d}\sigma \neq 0$ (like example in Jason's talk)
In general, the G_2-compatible ACMS is *not* contact.

G_2, ACMS and $SU(3)$ structures

R and $SU(3)$ structure ,...*Friedrich et.al:97*

Since they admit ACMS, (Y, φ) admit *two* nowhere-vanishing spinors $(\eta, R\eta)$.

This reduces the structure group of Y to $SU(3)$.

Remark:

For the G_2 connection ∇_T, we have $\nabla_T \eta = 0$, however generally $\nabla_T(R\eta) \neq 0$.
So, generally, $\mathrm{Hol}(\nabla_T) \subseteq G_2$.

Only when ACMS compatible with ∇_T: $\mathrm{Hol}(\nabla_T) \subseteq SU(3)$.

G_2, ACMS and $SU(3)$ structures

Transverse geometry and $SU(3)$ structure

ACMS $\implies$ (Y, φ) is foliated by 1D leaves (integral curves of R):
Locall, cone-like metric $\mathrm{d}s^2_\varphi = \sigma^2 + \mathrm{d}s^2_\perp$.
Vectors and k-forms can be decomposed w.r.t. the contact form σ:

- $u \in \Gamma(TY)$ is *transverse* to foliation $\mathcal{F}_R$ if $u \in \Gamma(\mathrm{Ker}(\sigma))$
- k-form α is transverse if $i_R(\alpha) = 0$
- $\forall$ k-form α, $\alpha = \sigma \wedge \alpha_0 + \alpha_\perp$, with α_0 and $\alpha_\perp$ transverse.

This transverse geometry has $SU(3)$ structure (ω, Ω), and

$$\varphi = \sigma \wedge \omega + \Omega_+ \ , \ \psi = *_\varphi \varphi = -\sigma \wedge \Omega_- + \frac{1}{2}\omega \wedge \omega \ .$$

This does not say there is a 6D submanifold.
(Y, φ) admits 6D leaves $\iff$ $\mathrm{Ker}(\sigma)$ integrable
$\iff$ $(\mathrm{d}\sigma)_\perp = 0$ $\iff$ ACS is *not* contact.

G_2, ACMS and $SU(3)$ structures

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G_2, ACMS and $SU(3)$ structures

Summary
Any G_2 structure manifold (Y, φ) admits (at least) an *ACMS* $(J, R, \sigma, g_\varphi)$.
Consequently, (Y, φ) has $SU(3)$ structure (ω, Ω).

Does this have an impact on string compactifications?

Let's look at 3D $\mathcal{N} = 1$ heterotic string vacua.

cf. talks by Jason, Mateo, Xenia

3D $\mathcal{N} = 1$ heterotic string compactifications

Heterotic compactifications $\mathcal{M}_{10} = \mathcal{M}_3 \times Y$

We have the following mathematical objects

- 7-dim Riemannian manifold Y with metric g_{mn}
- Vector bundle $V \to Y$ with connection A and structure group $G \subset E_8 \times E_8$
- Scalar ϕ
- 3-form flux defined by $H = \mathrm{d}B + \frac{\alpha'}{4}(CS(A) - CS(\Theta))$
  where Θ is a connection of TY

Want to study heterotic $\mathcal{N} = 1$ G_2 systems $[(Y, \varphi), (V, A), (TY, \Theta), H]$.

3D $\mathcal{N} = 1$ heterotic string compactifications

Heterotic $\mathcal{N} = 1$ G_2 system $[(Y, \varphi), (V, A), (TY, \Theta), H]$

SUSY constrains the geometry:

- $\exists$ spinor η on Y, nowhere vanishing, Killing: $\boxed{\nabla_H \eta = 0}$

  $\iff$ Y has integrable G_2 structure φ with torsion $T = H$.

  Integrable G_2 structure means $\tau_2 = 0$ *Fernandez, Ugarte:98*

- connections A and Θ are G_2 instantons: e.g. $\boxed{F(A) \wedge \psi = 0}$.
- Bianchi identity of anomaly cancellation condition links everything:

$$\mathrm{d}H = \frac{\alpha'}{4}(\mathrm{tr}F(A) \wedge F(A) - \mathrm{tr}R(\Theta) \wedge R(\Theta))$$

Proven that SUSY + BI $\implies$ heterotic EOM $\iff$ Θ is a G_2 instanton

Hull:86, Ivanov:10, Martelli–Sparks:10

3D $\mathcal{N} = 1$ heterotic string compactifications

Long history of heterotic $\mathcal{N} = 1$ G_2 systems

Günyadin, Nicolai:95, Gauntlett, Martelli, Waldram, Kim:01,Friedrich, Ivanov:01,03, Gauntlett, Martelli, Waldram:04,Ivanov, Ivanov:05, ...

Examples

- Compact G_2 holonomy manifold, w.o. flux, bundle from standard embedding
  Font:10
- Compact 7-fold with G_2 structure, w. flux, G_2 instanton bundle
  Fernandez, Ivanov, Ugarte, Villacampa:11, F, I, U, Vassiliev:15, Lotay, Sa Earp:21, de la Ossa, Galdeano:21 → Jason's and Mateo's talks
- Non-compact 7-fold with G_2 structure, w. flux, G_2 instanton bundle
  Günaydin, Nikolai:95; Fernandez, Ivanov, Ugarte, Villacampa:15; Hinoue, Yasui:14

Recent work on infinitesimal deformations of such systems

Garcia-Fernandez, Rubio, Tipler: 15, de la Ossa, ML, Svanes:16,17,19, Fiset, Quigley, Svanes:17, de la Ossa, ML, Magill, Svanes:19, Clarke, Garcia-Fernandez, Tipler:20, ... → Xenia's talk

ACMS in heterotic string compactifications

KSE+BI+ACMS $\rightarrow$ $SU(3)$ torsion, connections and flux of transverse geometry. In general, $\nabla_T \omega \neq 0$ and $\nabla_T \Omega \neq 0$ as **holonomy stays** G_2.

ACMS in heterotic string compactifications

Get $SU(3)$ **holonomy** whenever $\boxed{\nabla_T(R\eta) = 0}$

$$\iff \nabla_T R = 0 \iff \nabla_a \sigma_b = 0 \iff R \text{ is a Killing vector, and } \mathrm{d}\sigma = T_0$$

(recall $\alpha = \sigma \wedge \alpha_0 + \alpha_\perp$) *cf. Gran, Papadopoulos et. al. 05, 07,16*

- transverse geometry has Strominger-Hull $SU(3)$ structure.
- Y is $U(1)$ principal bundle over transverse geometry with hol. connection Σ (where $\sigma = \mathrm{d}r + \Sigma$, and $\mathrm{d}\sigma = \mathrm{d}\Sigma = \mathrm{d}_\perp\Sigma$ primitive).
- 3D spacetime is Minkowski.

This does not rely on integrability of $\mathrm{Ker}(\sigma)$ (i.e. $(\mathrm{d}\sigma)_\perp \neq 0$ in general)

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G_2, ACMS and $SU(3)$ structures

Example 1 (Y, φ) has G_2 holonomy:

$T = 0 \implies \mathrm{d}\sigma = 0$ and $\mathrm{Ker}(\sigma)$ is necessarily integrable.

Y is a codimension one foliation with leaves which are Calabi–Yau 3-folds.

G_2, ACMS and $SU(3)$ structures

Example 2 Nilmanifold $N(3,1)$ example *Fernandez, Ivanov, Ugarte, Villacampa:11*

Let e^a, $a = 1, .., 7$ be left-invariant 1-forms on $H(3,1)$; all closed but e^7

$$\mathrm{d}e^7 = ae^{12} + be^{34} + ce^{56} \, .$$

When $c = -(a+b)$ this geometry solves the $\mathcal{N} = 1$ heterotic G_2 system with

$$\varphi = (e^{12} + e^{34} + e^{56}) \wedge e^7 + e^{135} - e^{146} - e^{236} - e^{245}$$
$$T = -\tau_3 = -(a+b)e^{567} + be^{347} + ae^{127}$$
$$\kappa^A, \kappa^\Theta \sim e^7 \quad \text{both } G_2 \text{ instantons.}$$

A natural ACMS is given by $R = E_7$, $\sigma = e^7$. This gives

- $\mathrm{d}\sigma$ is purely transverse, so $\mathrm{Ker}(\sigma)$ is non-integrable.
- $\sigma \wedge \mathrm{d}\sigma \wedge \mathrm{d}\sigma \wedge \mathrm{d}\sigma \neq 0$: contact structure.
- $\mathrm{d}\sigma = T_0$, $\mathrm{d}\sigma \wedge \omega \wedge \omega = 0$: $\boxed{\mathcal{N} = 2 \text{ SUSY}}$.

G_2, ACMS and $SU(3)$ structures

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- $\mathrm{d}\sigma = T_0$, $\mathrm{d}\sigma \wedge \omega \wedge \omega = 0$: $\boxed{\mathcal{N} = 2 \text{ SUSY}}$.

G_2, ACMS and $SU(3)$ structures

Conclude: the ACMS can be a useful diagnostics of “hidden” SUSY (with torsion)

What about other heterotic G_2 solutions? Can readily check

- contact CY 7-manifolds of Lotay–Sa Earp
- Sasakian manifolds of de la Ossa–Galdeano

preserve exactly $\mathcal{N} = 1$ SUSY, not $\mathcal{N} = 2$.

G_2, ACM3S and $SU(2)$ structures

The nilmanifold example had multiple ACMS (e.g. 7 left-invariant ones).

How many ACMS are guaranteed on general 7D G_2 structure manifolds?

Thomas:69

Any compact, orientable 7-dimensional manifold Y admits two globally defined, everywhere linearly independent vector fields $R^1, R^2 \in \Gamma(TY)$.

WLOG (Y, φ) admit an orthonormal 2-frame (R^1, R^2), and thus two ACMS that are compatible with the G_2 metric.

G_2, ACM3S and $SU(2)$ structures

Kuo:70

If the ACMSs associated to (R^1, R^2) satisfy

$$\begin{aligned}
&\sigma^1(R^2) = \sigma^2(R^1) = 0 \\
&J^1(R^2) = -J^2(R^1) \\
&\sigma^1 \circ J^2 = -\sigma^2 \circ J^1 \\
&J^1 J^2 - R^1 \otimes \sigma^2 = -J^2 J^1 + R^2 \otimes \sigma^1 \ ,
\end{aligned}$$

then Y admits a third ACMS $J^3 = J^1 J^2 - R^1 \otimes \sigma^2$, $R^3 = J^1(R^2)$, $\sigma^3 = \sigma^1 \circ J^2$. Together, these three ACMS define an almost contact metric 3-structure (ACM3S)

G_2 structure manifold: *Todd:15*
ACMSs associated to (R^1, R^2) automatically satisfy Kuo's constraints.

G_2, ACM3S and $SU(2)$ structures *Kuo:70,..,Arikan et.al:11,12, Todd:15*

G_2 structure manifold (Y, φ): ON 2-frame (R^1, R^2) gives a 3rd vector:

$$R^3 = R^1 \times_{\varphi} R^2 \ .$$

Thus, any (Y, φ) admits ACM3S specified by (R^1, R^2, R^3), s.t.

$$\sigma^{\alpha}(R^{\beta}) = \delta^{\alpha\beta} \ \ , \ \ J^{\alpha}(R^{\beta}) = \epsilon^{\alpha\beta\gamma} R^{\gamma} \ \ , \ \ \sigma^{\alpha} \circ J^{\beta} = -\sigma^{\beta} \circ J^{\alpha} \ .$$

In addition,

$$\sigma^{\gamma} = \frac{1}{2} \epsilon^{\alpha\beta\gamma} i_{R^{\beta}} i_{R^{\alpha}} \varphi \ .$$

This implies, for the G_2 3-form

$$\varphi = \frac{1}{3!} \epsilon_{\alpha\beta\gamma} \sigma^{\alpha} \wedge \sigma^{\beta} \wedge \sigma^{\gamma} + \sum_{\alpha} \sigma^{\alpha} \wedge \omega_{\perp}^{\alpha} \ .$$

G_2, ACM3S and $SU(2)$ structures *Kuo:70,..,Arikan et.al:11,12, Todd:15*

The ACM3S gives

$$\varphi = \frac{1}{3!}\epsilon_{\alpha\beta\gamma}\sigma^{\alpha} \wedge \sigma^{\beta} \wedge \sigma^{\gamma} + \sum_{\alpha} \sigma^{\alpha} \wedge \omega_{\perp}^{\alpha} \; .$$

Selecting (R^1, R^2, R^3) induces a decomposition of the tangent bundle

$$TY \cong \mathcal{T} \oplus \mathcal{T}^{\perp},$$

This reduces the structure group to $SU(2)$ (i.e. subgroup that preserves 3-frame).

G_2, ACM3S and $SU(2)$ structures

Comment $SU(2)$ and $SU(3)$ structures have been used in constructing and classifying $\mathcal{N} = 1$ string vacua.

Behrnt, Cvetic, Liu:05, Kim:05, Gran, Gutowski, Roest:07, Andriolo, Shiu, Triendl, van Riet, Venken, Zoccarato:18, Passias, Prins:19

ACM3S, integrability and calibrated cycles

Recall, for ACS, R decomposes $TY \cong \mathcal{T} \oplus \mathcal{T}^{\perp}$.

- $\mathcal{T}$ is trivially involutive; tangent to 1D leaves of foliation.
- $\mathcal{T}^{\perp}$ involutive if $\mathrm{d}\Sigma$ transverse.

For ACM3S, (R^1, R^2, R^3) also induces a decomposition $TY \cong \mathcal{T} \oplus \mathcal{T}^{\perp}$.

- $\mathcal{T}$ $(\mathcal{T}^{\perp})$ need not be tangent to a 3D (4D) foliation.

ACM3S, integrability and calibrated cycles

$\mathcal{T}$ involutive $\iff$ $i_{R^\alpha} i_{R^\beta} \mathrm{d}\xi = 0$ for all $\xi \in \Gamma(\mathcal{T}^{*,\perp}), R^\alpha \in \Gamma(\mathcal{T})$.

- $\boxed{\mathcal{T} \text{ is involutive:}}$ $\mathcal{T}_x$ is tangent to a leaf $\mathcal{L}_x$
- $\mathcal{L}$ is parallelisable with volume form $\mathrm{Vol}_{\mathcal{L}} = \mathrm{d}\sigma^{123}$.
  Recall every 3D oriented manifold is parallelisable.
- In fact $\mathrm{Vol}_{\mathcal{L}} = \mathrm{d}\sigma^{123} = \varphi|_{\mathcal{L}} \implies$ $\boxed{\mathcal{L} \text{ is an associative cycle}}$
- If $\mathrm{d}\varphi = 0$, $\mathcal{L}$ is also calibrated (... so branes wrapping $\mathcal{L}$ give BPS states).

Similarly, $\mathcal{T}^\perp$ is involutive $\implies$ Y is foliated by co-associative 4-cycles.
($\mathcal{T}^\perp$ is involutive if and only if $\mathrm{d}\sigma^\alpha = \sigma^\beta \wedge \mu^\alpha_\beta$, for one-forms μ^α_β).

ACM3S, integrability and calibrated cycles

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Similarily, $\mathcal{T}^\perp$ is involutive $\implies$ Y is foliated by co-associative 4-cycles.
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ACM3S, integrability and calibrated cycles

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- In fact $\mathrm{Vol}_{\mathcal{L}} = \mathrm{d}\sigma^{123} = \varphi|_{\mathcal{L}} \implies$ $\boxed{\mathcal{L} \text{ is an associative cycle}}$
- If $\mathrm{d}\varphi = 0$, $\mathcal{L}$ is also calibrated (... so branes wrapping $\mathcal{L}$ give BPS states).

Similarily, $\mathcal{T}^\perp$ is involutive $\implies$ Y is foliated by co-associative 4-cycles.
($\mathcal{T}^\perp$ is involutive if and only if $\mathrm{d}\sigma^\alpha = \sigma^\beta \wedge \mu^\alpha_\beta$, for one-forms μ^α_β).

ACM3S, integrability and calibrated cycles

$\mathcal{T}$ involutive $\iff$ $i_{R^\alpha} i_{R^\beta} \mathrm{d}\xi = 0$ for all $\xi \in \Gamma(\mathcal{T}^{*,\perp}), R^\alpha \in \Gamma(\mathcal{T})$.

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Space of ACM3Ss

Natural question:

is there a non-trivial space $\mathscr{C}$ of ACM3 structures compatible with a given G_2 structure?

Recall:
ACM3S is uniquely given by ON 3-frame (R^1, R^2, R^3), where $R^3 = R^1 \times_\varphi R^2$.

So fiberwise, need space of all ON, ordered pairs of vector fields in $T_xY \cong \mathbb{R}^7$:
$\implies$ the Stiefel manifold $V_2(T_xY) \cong G_2/SU(2)$ *Harvey, Lawson:82*.

Globally, get fibre bundle $\mathcal{V}_2(TY)$ with typical fibre $V_2(\mathbb{R}^7)$, and ON 2-frames (e.g. ACM3S) as sections.

Thus, the space of ACM3S is given by $\mathscr{C} = \Gamma(Y, \mathcal{V}_2(TY))$.

Comment: this is a non-empty space (Thomas:69)

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$$\mathscr{T} \to \mathscr{C} \xrightarrow{\pi} \mathscr{S}$$

with base corresponds to the space of splittings $\mathscr{S}$ of the form $T^*Y = \mathcal{T}^* \oplus \mathcal{T}^{*\perp}$.

The space of splitting choices at a given point is $G(\varphi_x) = G_2/SO(4)$

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Thus, let $\mathcal{G}(\varphi) \to Y$ be fibre bundle with typical fibre $G(\varphi_0)$.
Then $\tilde{\mathscr{S}} := \Gamma(Y, \mathcal{G}(\varphi))$ contains $\mathscr{S}$.

Given $\mathscr{S}$, find $\mathscr{T}$ by first fixing an arbitrary initial ON 3-frame (R^1, R^2, R^3) with $R^3 = R^1 \times_\varphi R^2$. By $SO(3)$ transformations we can reach any other trivialisation (S^1, S^2, S^3) satisfying the ACM3S constraints. $\rightsquigarrow \mathscr{T} \sim \mathrm{Maps}(Y, SO(3))$.

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Space of ACM3Ss

Example

Space of ACM3S in local neighbourhood of an associative three-cycle $X \in Y$.

- Construct ACM3S which restricts to a trivialisation of TX.
- Fixing boundary conditions is more subtle: topologically distinct ACMS at the edge of the local region.
- For $X = S^3$ we prove $\mathscr{T} \to \mathscr{C} \xrightarrow{\pi} \mathscr{S}$ is a non-trivial fibration.

Conclusions and outlook

Conclusions

Manifolds with G_2 structure (Y, φ) admit almost contact 3-structures

- $\chi(Y) = 0 \implies$ nowhere-vanishing vector field R and 1D foliation
- In fact, any spin 7D manifold admits an ON 2-frame (R^1, R^2)
- Using φ find a 3rd vector: ON 3-frame (R^1, R^2, R^3) with $R^3 = R^1 \times_\varphi R^2$.
- (R^1, R^2, R^3) equips (Y, φ) with an ACM3S.

ACM(3)S $\implies$ splitting of tangent bundle $TY \cong \mathcal{T} \oplus \mathcal{T}^\perp$

- ACMS: 1D foliation guaranteed, 6D foliation only if $(\mathrm{d}\sigma)_\perp = 0$.
- ACM3S: 3D/4D $\mathcal{T}/\mathcal{T}^\perp$ not necessarily involutive.
  When they are, Y is foliated by associatives/coassociatives.

Conclusions and outlook

Conclusions

Manifolds with G_2 structure (Y, φ) admit ACM3S specified by (R^1, R^2, R^3)

$$\varphi = \frac{1}{3!}\epsilon_{\alpha\beta\gamma}\sigma^\alpha \wedge \sigma^\beta \wedge \sigma^\gamma + \sum_\alpha \sigma^\alpha \wedge \omega_\perp^\alpha \, ,$$

ACM(3)S $\implies$ splitting of tangent bundle $TY \cong \mathcal{T} \oplus \mathcal{T}^\perp$

Outlook

- Probe relation between ACM(3)S and SUSY for type II, M theory.
- Effect of ACM(3)S on 3D/4D EFT arising from string compactifications?
  Action (superpotential); parametrisation & metric of moduli space?
- Determine space of ACM3S (in examples)
  Interesting that $\mathscr{C}$ is topological, e.g. for compact (Y, φ) with G_2 holonomy?
- Strengthen link between $\mathcal{T} = \mathcal{T} \oplus \mathcal{T}^\perp$, involutivity and calibrated manifolds.

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Manifolds with G_2 structure (Y, φ) admit ACM3S specified by (R^1, R^2, R^3)

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Thank you for listening!

Additional slides: 3D EFT, G_2 and ACS

Moduli space as critical locus of a superpotential on off-shell parameter space

Strategy: Dimensional reduction $\Longrightarrow$ 3D gravitino mass $\boxed{M_{3/2} = e^K W}$

Remark: need Hessian K

de la Ossa, ML, Magill, Svanes:19

$$W = \frac{1}{4} \int_Y e^{-2\phi} \left((H + h\,\varphi) \wedge \psi - \frac{1}{2} \mathrm{d}\varphi \wedge \varphi \right)$$

Can show $\delta W = 0 \iff \mathcal{N} = 1$ heterotic G_2 system.
Caveat: 3D $\mathcal{N} = 1$ supergravity lacks non-renormalisation theorems.

Additional slides: Superpotential and dimensional reduction

Fermionic part of 10D heterotic supergravity action

Bergshoeff, de Roo:89, Gurrieri, Lukas, Micu:07

$\rightarrow$ 3D kinetic and mass terms for gravitino

$$S_{0,f} = -\frac{1}{2\kappa_{10}^2} \int_{M^{10}} d^{10}x\sqrt{-g}\, e^{-2\phi} \left(\overline{\Psi}_M \Gamma^{MNP} D_N \Psi_P - \frac{1}{24} \left(\overline{\Psi}_M \Gamma^{MNPQR} \Psi_R + 6\, \overline{\Psi}^N \Gamma^P \Psi^Q \right) H_{NPQ} \right)$$

$$S_{3D} \supset -\frac{1}{2\kappa_3^2} \int d^3x \sqrt{-g} \left(\overline{\psi}_\mu \Gamma^{\mu\nu\kappa} D_\nu \psi_\kappa + m \overline{\psi}_\mu \Gamma^{\mu\kappa} \psi_\kappa \right)$$

Straightforward, but

- field normalisation (correct EH term)
- conventions for gravitino mass in AdS (want SUSY $\leftrightarrow 0 = M_{3/2} \sim W$).

Additional slides: Superpotential and dimensional reduction

10D action contains $\int d^{10}X \sqrt{-g_{10}}\, e^{-2\phi} \left(-\frac{1}{24}\overline{\Psi}_\mu \Gamma^{\nu\mu}\Gamma^{ijk}\Psi_\nu H_{ijk}\right)$:

$\rightsquigarrow$ gravitino mass contributions

$$M_{3/2} = \frac{1}{4}\int_7 e^{-2\phi+n}\left(-\frac{1}{2}d\varphi\wedge\varphi + (H + h\varphi)\wedge\psi\right) .$$

(fixing conventions so that SUSY $\implies$ $M_{3/2} = 0$)
Analysing the Einstein–Hilbert term $\implies$

$$W = \frac{1}{4}\int_Y e^{-2\phi}\ \left((H + h\,\varphi)\wedge\psi - \frac{1}{2}\,\mathrm{d}\varphi\wedge\varphi\right)$$

Can show $\delta W = 0 \quad \iff \mathcal{N} = 1$ heterotic G_2 system

Additional slides: ACS and moduli space geometry

A problem in analysing beyond infinitesimal deformations is the lack a natural parametrisation for the parameter space.
Phrased differently: as 3D superpotential is real, lack the guide from holomorphy

ACS $\rightsquigarrow$ transverse geometry is (almost) complex $\rightsquigarrow$ parametrisation?
Hint: superpotential decomposes

$$W = \int_Y e^{-2\phi}\,\sigma \wedge \mathrm{Im}\Big(\Big[\hat{H} + i\,\mathrm{d}_\perp\omega + \frac{1}{8}\left(\omega\lrcorner(H_0 - \mathrm{d}_\perp\Sigma) + 7\,h - \left(\Sigma\wedge H_0 + \frac{i}{2}R(\Omega_+)\right)\lrcorner\bar{\Omega}\right)\bar{\Omega}\Big]\wedge\Omega\Big)\ ,$$

where

$$\hat{H} = \mathrm{d}\hat{b} + \frac{\alpha'}{4}\left(\mathcal{CS}(\hat{a}) - \mathcal{CS}(\hat{\theta})\right)\ .$$