Non-Kähler Calabi-Yau Geometry and Pluriclosed Flow

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Outline

Background

Hermitian Geometry

Definition 1 (Hermitian Geometry - Basic Definitions)

Given a Hermitian manifold (M^{2n}, J, g), we make the following definitions.

- $\omega = gJ$ (and often g) is called pluriclosed if $\sqrt{-1}\partial\overline{\partial}\omega = 0$.
- A metric-compatible connection ∇ on a Hermitian bundle is called Hermitian if $\nabla J \equiv 0$.
- Provided ∇ is Hermitian and the bundle holomorphic as well, we will call ∇ Chern if $\nabla^{0,1} = \overline{\partial}$ (on $T_{\mathbb{C}}M$ equiv. torsion has no $(1,1)$ part).
- Provided ∇ is a Hermitian connection on $T_{\mathbb{C}}M$, we call ∇ Bismut if its torsion can be lowered to a totally anti-symmetric 3-form.

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- It can be shown that the Chern and Bismut connections are unique when g is pluriclosed with torsions depending on $\partial\omega$ and $\overline{\partial}\omega$. When $d\omega = 0$, these collapse into the Levi-Civita connection.

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- Gauduchon ('97) showed that a complex manifold with a pluriclosed metric admits a distinguished, one-parameter family of Hermitian connections determined uniquely by the Chern and Bismut connections.

Hermitian Geometry

Definition 2 (Curvature tensor, First and Second Chern-Ricci Curvature, First Bismut-Ricci Curvature)

On a pluriclosed, Hermitian manifold (M^{2n}, J, g) we denote by Ω^C and Ω^B the curvatures of the Chern and Bismut connections on $T_{\mathbb{C}}M$ respectively.

$$g\Omega^C \in \Lambda^{1,1} \otimes \Lambda^{1,1} \qquad g\Omega^B \in \Lambda^2 \otimes \Lambda^{1,1}$$

$$\rho^C = \operatorname{tr} \Omega^C \in \Lambda^{1,1} \qquad \rho^B = \operatorname{tr} \Omega^B \in \Lambda^2$$

$$S = \sqrt{-1}\Lambda_\omega \Omega^C \in \operatorname{End}(TM \otimes \mathbb{C}) \qquad \text{(not relevant in this paper)}$$

Holomorphic Courant Algebroids - Basic Definitions

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Definition 3 (Exact Holomorphic Courant Algebroid)

Given a complex manifold (M, J), $\tau \in \Lambda^{3,0+2,1}$, $d\tau = 0$, we define the exact holomorphic Courant algebroid $\mathcal{Q}_\tau = (T^{1,0} \oplus T^*_{1,0}, \partial_\tau, \langle , \rangle, [,]_\tau, \pi)$ where:

- $\overline{\partial}_\tau(X + \xi) = \overline{\partial}X + \overline{\partial}\xi - \sqrt{-1}\tau^{2,1}$,
- $\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\eta(X) + \xi(Y))$,
- $[X + \xi, Y + \eta]_\tau = [X, Y] + \partial(\eta(X)) + i_X\partial\eta - i_Y\partial\xi + i_Y i_X\tau^{3,0}$,
- $\pi : T^{1,0} \oplus T^*_{1,0} \to T^{1,0}$ such that $\pi(X + \xi) = X$.

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- In the pluriclosed setting, $\tau = 2\sqrt{-1}\partial\omega$.

Holomorphic Courant Algebroids - Generalized Metrics

Definition 4

We call G a generalized Hermitian metric on $\mathcal{Q}_{\tau_0}$ provided G is a Hermitian metric on $T^{1,0} \oplus T^*_{1,0}$ of the form

$$G(X+\xi, \overline{X+\xi}) = g(X, \overline{X}) + g^{-1}(i_X\beta, i_{\overline{X}}\overline{\beta}) - 2\,\mathrm{Im}(g^{-1}(i_X\beta, \overline{\xi})) + g^{-1}(\xi, \overline{\xi}).$$

where g is a pluriclosed Hermitian metric on $T^{1,0}$ and $\beta \in \Lambda^{2,0}$ satisfies $\tau_0 - \partial\omega_g = d\beta$.

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- Only require $\tau_0 - \partial\omega = \overline{\partial}\beta$ corresponding to a loosening of equivalence from isomorphic as holo'c Courant algebroids to isomorphic as holomorphic, orthogonal bundles.

Bismut Hermitian-Einstein

Bismut Hermitian-Einstein Metrics

Definition 5

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$$\rho^B(g) := \sqrt{-1}\,\mathrm{tr}\,\Omega^B_g \equiv 0$$

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- ρ^B is a representative of $c_1(M)$.
- Reduces to $\rho^{LC} = 0$ in Kähler setting.
- $\rho^B = \rho^C + dd^*\omega$

Holomorphic Courant Algebroids - Bismut's Identity

In a 1989 paper, JM Bismut proved an identity between the Chern curvature of $T^{1,0} \oplus T^*_{1,0}$ with a twisted holo'c structure and the Bismut curvature of $T_{\mathbb{C}}M$.

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Theorem 6 ('21 Garcia-Fernandez, J, Streets; '89 Bismut)

On (M, J, ω_0) pluriclosed and given $\omega \in \Lambda^{1,1}_{\mathbb{R}}$ pluriclosed and $\beta \in \Lambda^{2,0}$ s.t. $\overline{\partial}\beta = \partial\omega_0 - \partial\omega$. From this, we can define $\mathcal{Q}_{\sqrt{-1}\partial\omega}$ and $G(\omega, 0)$ as above. We also define an isomorphism of vector bundles.

$$\psi_g : T_{\mathbb{C}}M \to T^{1,0} \oplus T^*_{1,0}, \quad X = X^{1,0} \oplus X^{0,1} \mapsto X^{1,0} - g(X^{0,1})$$

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- *Define a Hermitian connection on $T_{\mathbb{C}}M$ as follows.*

$$g(\nabla^-_X Y, Z) = g(\nabla^{LC,g}_X Y, Z) + \frac{1}{2} d^c\omega(X, Y, Z)$$

This defines a holo'c structure on $T_{\mathbb{C}}M$ by $(\nabla^-)^{0,1}$ and ψ_g can be viewed as a holo'c isometry $\psi_g : (T_{\mathbb{C}}M, (\nabla^-)^{0,1}) \to \mathcal{Q}_{\sqrt{-1}\partial\omega}$.

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- *Denote the Chern connection on the holo'c, Herm. vector bundle $(\mathcal{Q}_{\sqrt{-1}\partial\omega}, G)$ by ∇^G. Further, $\nabla^G = (\psi_g)_* \nabla^-$, $\Omega^G = (\psi_g)_* \Omega^-$.*

Bismut Hermitian Einstein and Slope Stability of $\mathcal{Q}$

Theorem 7 ('21 Garcia-Fernandez, J, Streets)

Given a pluriclosed manifold (M, J, g_0) and considering the holo'c Herm. vector bundle associated to $\mathcal{Q}_{\sqrt{-1}\partial\omega_0}$, we define a generalized Herm. metric $G = G(\omega, \beta)$. Then the Chern connection ∇^G on $\mathcal{Q}_{\sqrt{-1}\partial\omega_0}$ satisfies the following equation of endomorphisms of the underlying vector bundle of $\mathcal{Q}_{\sqrt{-1}\partial\omega_0}$.

$$S_g^G := \sqrt{-1}\Lambda_\omega \Omega^G = \sqrt{-1}(e^{\sqrt{-1}\beta})^* \begin{pmatrix} -g^{-1}\rho_B^{1,1} & g^{-1}\rho_B^{0,2}g^{-1} \\ \rho_B^{2,0} & -\rho_B^{1,1}g^{-1} \end{pmatrix}$$

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- This follows from the identity $\Omega^B(X, Y, Z, W) = \Omega^-(Z, W, X, Y)$ and tracing Bismut's identity $\Omega^{C,G} = (\psi_g)_*\Omega^-$, it can also be verified directly by a brutal calculation.

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- This statement implies that g is Bismut Hermitian-Einstein iff $S_g^G \equiv 0$.

Bismut Hermitian Einstein and Slope Stability of $\mathcal{Q}$

Theorem 8 ('21 Garcia-Fernandez, J, Streets)

(M, J, g) *a pluriclosed Hermitian manifold. Let* $a = [\tilde{\omega}^{n-1}] \in H_A^{n-1,n-1}$, *where* $\tilde{\omega}$ *is the unique Gauduchon metric in the conformal class of* ω *s.t.* $\int_M \omega^n = \int_M \tilde{\omega}^n$. *Let* $\mathcal{Q} = \mathcal{Q}_{\sqrt{-1}\partial\omega}$ *as before. If* g *is Bismut Hermitian-Einstein, then for every subsheaf* $\mathcal{F} \subset \mathcal{Q}$ *we must have*

$$\mu_a(\mathcal{F}) = \frac{c_1(\det \mathcal{F}) \cdot a}{r} \leq 0.$$

Equality is obtained iff $\mathcal{Q}$ *splits holo'cally.*

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- This result is immediate because $c_1(\mathcal{Q}) = 0 \in H_{BC}^{1,1}(M)$, so $\mu_a(\mathcal{Q}) = 0$, and the result follows from Donaldson-Uhlenbeck-Yau.

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- Notice $\omega \mapsto [\tilde{\omega}^{n-1}]_{Aep}$ is complicated in general, but when $n = 2$, the Gauduchon criterion $\sqrt{-1}\partial\bar{\partial}\tilde{\omega}^{n-1} = 0$ is equivalent to the pluriclosed condition.

Main Theorem - The Obstruction

In what follows, we call $a \in H_A^{n-1,n-1}(M)$ positive if there is a Gauduchon metric $\tilde{\omega}$ s.t. $a = [\tilde{\omega}^{n-1}]_A$.

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Theorem 9 ('21 Garcia-Fernandez, J, Streets)

(M, J) connected, compact, complex manifold admitting a pluriclosed, Bismut Hermitian-Einstein metric g. Then there exists a positive Aeppli class $a \in H_A^{n-1,n-1}$ s.t., for any complex manifold Z and any map $f : M \to Z$ with df surjective at one point, one has

$$f^* c_1(Z) \cdot a \geq 0.$$

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Corollary 10

Taking the setup of the prior theorem, if Z has $c_1(Z) < 0$, then M cannot admit a pluriclosed, Bismut Hermitian-Einstein metric.

Implications for Pluriclosed Flow

Pluriclosed Flow

Definition 11

Given (M, J) a complex manifold, a one-parameter family $(\omega_t, \beta_t) \in \Lambda^{1,1}_{\mathbb{R}} \oplus \Lambda^{2,0}$ is said to be a solution to pluriclosed flow provided

$$\frac{\partial \omega_t}{\partial t} = -\rho_B^{1,1}(\omega_t), \qquad \frac{\partial \beta_t}{\partial t} = -\rho_B^{2,0}(\omega_t)$$

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Theorem 12 ('21 Garcia-Fernandez, J, Streets)

Given $(M^{2n}, J, \omega_t, \beta_t)$ a solution to pluriclosed flow, the associated family of generalized Hermitian metrics $G_t = G(\omega_t, \beta_t)$ on $\mathcal{Q}_{\sqrt{-1}\partial\omega_0}$ satisfies

$$G^{-1}\frac{\partial G}{\partial t} = -S_g^G.$$

Schwarz Lemma

Theorem 13 (Classical Schwarz Lemma)

Let $(\mathcal{E}, \overline{\partial}, h) \to (M, J, g)$ be a holo'c Hermitian vector bundle over a Hermitian manifold. Then if we have a holo'c section $\sigma \in H^0(M, \mathcal{E})$, it must be that

$$\Delta_g^C |\sigma|_h^2 = |\nabla_h^C \sigma|_{g,h}^2 + h(S_g^h \sigma, \sigma)$$

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Corollary 14 ('21 Garcia-Fernandez, J, Streets)

Let $\mathcal{E} \to (M, J, g)$ be a holo'c vector bundle over a Hermitian manifold. If $G, \tilde{G}$ are Hermitian metrics on $\mathcal{E}$, then

$$\Delta_g^C \operatorname{tr}_G \tilde{G} = |\Upsilon(G, \tilde{G})|_{G^{-1}, \tilde{G}}^2 + \operatorname{tr}_G \tilde{G}((S_g^G - S_g^{\tilde{G}}) \cdot, \cdot).$$

We use $\Upsilon(G, \tilde{G})$ to mean the difference of the G and $\tilde{G}$ Chern connections on $\mathcal{E}$.

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We use $\Upsilon(G, \tilde{G})$ to mean the difference of the G and $\tilde{G}$ Chern connections on $\mathcal{E}$. Moreover, if $G_t = G(\omega_t, \beta_t)$ for a solution to pluriclosed flow, we get the following useful evolution equation.

$$\left(\frac{\partial}{\partial t} - \Delta_g^C\right) \operatorname{tr}_G \tilde{G} = -|\Upsilon(G, \tilde{G})|^2_{G^{-1},\tilde{G}} + \operatorname{tr}_G \tilde{G}(S_g^{\tilde{G}} \cdot, \cdot)$$

Main Theorem - Existence and Convergence on Bismut-flat

Theorem 15 ('21 Garcia-Fernandez, J, Streets)

(M^{2n}, J, ω_F) *compact, Bismut-flat Hermitian manifold. Given* ω_0 *pluriclosed with* $[\partial\omega_0] = [\partial\omega_F] \in H^{2,1}_{\bar{\partial}}$ *the solution to pluriclosed flow with initial metric* ω_0 *exists on* $[0, \infty)$ *and converges to a Bismut-flat metric* ω_∞.

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- C^0 estimate:

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- C^1 (similar to Calabi C^3) estimate:

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Main Theorem - Existence and Convergence on Bismut-flat

Theorem 15 ('21 Garcia-Fernandez, J, Streets)

(M^{2n}, J, ω_F) *compact, Bismut-flat Hermitian manifold. Given* ω_0 *pluriclosed with* $[\partial\omega_0] = [\partial\omega_F] \in H^{2,1}_{\bar\partial}$ *the solution to pluriclosed flow with initial metric* ω_0 *exists on* $[0, \infty)$ *and converges to a Bismut-flat metric* ω_∞*.*

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- A blow up argument then gives C^∞ regularity. Convergence and flatness follow from these estimates and an argument on subsequential limits.

Main Theorem - Moduli Space of Generalized Kähler Structures

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- This completely classifies all Bismut-flat structures on e.g. the standard Hopf surface.

Main Theorem - Global Existence on $\kappa \geq 0$

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Let (M^4, J) be a minimal, non-Kähler, complex surface of Kodaira dimension $\kappa \geq 0$. Given ω_0 a pluriclosed metric on M, the solution to pluriclosed flow exists on $[0, \infty)$.

- Suffices to check principal T^2-bundles over Riemann surfaces $\pi : M^4 \to \Sigma^2$ with genus ≥ 1 by Kodaira classification.

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- Earlier Schwarz Lemma to estimate $\mathrm{tr}_G \, \tilde{G}$: $\left(\frac{\partial}{\partial t} - \Delta_g \right) \mathrm{tr}_G \, \tilde{G} \leq C \, \mathrm{tr}_G \, \tilde{G}$.