Futaki invariants and harmonic metrics for the Hull-Strominger system

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ICMAT-CSIC
Generalized Geometry in Interaction

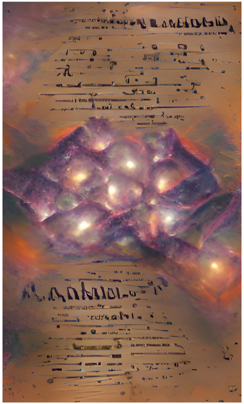
16th June 2022

I+D+i SEV2015-05554-18-1 funded by MCIN/AEI/10.13039/501100011033

Joint work with M. Garcia-Fernandez

Motivation

- Geometrization of *Reid's fantasy*.
- Conifold transitions connect CY 3-folds with different topology.
- Obstructions to pushing solutions along surgeries.



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Definition

The Hull-Strominger system for a hermitian metric ω on X and a pair of hermitian metrics h_i on V_i and $\alpha \in \mathbb{R}_{>0}$ consists of

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\begin{aligned}
F_{h_i} \wedge \omega^2 &= 0 \\
d(||\Omega||_\omega \omega^2) &= 0, \qquad \mathfrak{b} = [||\Omega||_\omega \omega^2] \in H^{2,2}_{BC}(X, \mathbb{R}) \\
dd^c\omega - \alpha \operatorname{tr} F_{h_1} \wedge F_{h_1} + \alpha \operatorname{tr} F_{h_2} \wedge F_{h_2} &= 0
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Sometimes: $V_0 \cong T^{1,0}$

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$\rightsquigarrow$ Strategy: Geometrize anomaly cancellation equation

String algebroids

Bismut's observation:

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Then, under the identification $Q \xrightarrow{(\mathrm{Id}, g^{-1})} T^{1,0} \oplus T^{0,1}$, $D^G \mapsto \nabla^-_{\mathbb{C}}$

$$\nabla^- = \nabla^g + \tfrac{1}{2} g^{-1} d^c\omega$$

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where $\langle \cdot, \cdot \rangle = -\alpha \operatorname{tr}_{V_0} + \alpha \operatorname{tr}_{V_1}$

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$$D^G_V(W + r_1 + r_2) = \nabla^-_V W - g^{-1}\langle i_V F_{h_i} r_i\rangle + d^{h_i} r_i - F_{h_i}(V, W)$$
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$\rightsquigarrow$ Observe that we recover Bismut's observation

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Upshot: Generalized geometry 'encodes' the information of a solution in $(Q, D^{\mathbf{G}})$.

Curvature of generalized hermitian metrics

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Curvature tensor of $\mathbf{G}$:

$$F_G = \begin{pmatrix} R_{\nabla^-} - \mathbb{F}^\dagger \wedge \mathbb{F} & -\mathbb{I}^\dagger \\ \mathbb{I} & [F_{h_i},] - \mathbb{F} \wedge \mathbb{F}^\dagger \end{pmatrix}$$

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where

$$\begin{aligned} i_W i_V \mathbb{F}^\dagger \wedge \mathbb{F}(Z) &= g^{-1}\langle i_W F_{h_i}, F_{h_i}(V, Z)\rangle - g^{-1}\langle i_V F_{h_i}, F_{h_i}(W, Z)\rangle, \\ i_W i_V \mathbb{I}(Z) &= (\nabla_Z^{h_i,-} F_{h_i})(V, W) - F_{h_i}(V, g^{-1} i_Z i_W d^c\omega) + F_{h_i}(W, g^{-1} i_Z i_V d^c\omega), \\ i_W i_V \mathbb{F} \wedge \mathbb{F}^\dagger(r) &= F_{h_i}(W, g^{-1}\langle i_V F_{h_i}, r\rangle) - F_{h_i}(V, g^{-1}\langle i_W F_{h_i}, r\rangle). \end{aligned}$$

Curvature of generalized hermitian metrics

Theorem (Garcia-Fernandez, GM)

Let (ω, h_0, h_1) be a solution to the Hull-Strominger system, and let $(Q, \mathbf{G})$ be its associated hol. Courant algebroid. Then

$$F_{\mathbf{G}} \wedge \omega^{n-1} = 0$$

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- Exact counterpart of result in perturbation theory by De la Ossa-Larfors- Svanes [*Restrictions of Heterotic G2 Structures and Instanton Connections, '17*] in the physics literature.
- Under strictly weaker assumptions, the same result holds $\rightsquigarrow$ *GRic* flatness, Gen. Ricci flow // Supergravity eqs.
- Expect a generalized geometric proof.

Futaki invariant for the Hull-Strominger system

Existence of Hermitian-Einstein metrics admits a moment map interpretation
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Proposition

Let X be compact complex, $\mathfrak{b} \in H^{n-1,n-1}_{BC}(X, \mathbb{R})$ a balanced class, and let $\mathcal{E} \to X$ be a holomorphic vector bundle. Then, the map

$$\mathcal{F}_{\mathfrak{b}} : H^0(X, \mathrm{End}\, \mathcal{E}) \to \mathbb{C}$$
$$\varphi \mapsto \int_X \mathrm{tr}(\varphi \circ F_h) \wedge \omega^{n-1}$$

defines a character that does not depend on the (pseudo)hermitian metric h nor on the representative $\omega^{n-1} \in \mathfrak{b}$. Moreover, $\mathcal{F}_{\mathfrak{b}}$ vanishes if there exist (ω_0, h_0) such that $\omega_0^{n-1} \in \mathfrak{b}$,

$$F_{h_0} \wedge \omega_0^{n-1} = 0$$

Futaki invariant for the Hull-Strominger system

To introduce the Futaki invariant in our setting:

Proposition (Garcia-Fernandez, Shahbazi, Rubio, Tipler)

Let (X, Ω) Calabi-Yau, holomorphic bundles V_0, V_1. Let $\mathfrak{S}$ be the set of equivalence classes of hol. Courant algebroids extending V_i. Then, $\mathfrak{S}$ is an affine space for the vector space given by the image of

$$H_A^{1,1}(X) \xrightarrow{\partial} H^1(\Omega^{2,0} \xrightarrow{d} \Omega^{3,0+2,1} \xrightarrow{d} \ldots)$$

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$\rightsquigarrow$ family of Futaki characters:

$$\mathfrak{H} \to \mathfrak{S}, \qquad \mathfrak{H}_{\mathfrak{s}} = H^0(X, \operatorname{End} Q_{\mathfrak{s}}), \qquad \mathfrak{s} \in \mathfrak{S}$$

$$\{\mathcal{F}^{\mathfrak{s}} : \mathfrak{H}_s \to H_A^{1,1}(X)\}_{\mathfrak{s}\in\mathfrak{S}}, \qquad \varphi \mapsto [\operatorname{tr}(\varphi \circ F_{\mathbf{G}})]$$

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Theorem (Garcia-Fernandez, GM)

Let (X, Ω) Calabi-Yau and let V_0, V_1 be holomorphic vector bundles, and let $\mathfrak{S}$ be the affine space of hol. Courant algebroids extending V_i. Suppose (ω, h_0, h_1) is a solution to the Hull-Strominger system. Then, there exists $\mathfrak{s} \in \mathfrak{S}$ such that

$$\langle \mathcal{F}^{\mathfrak{s}}, \mathfrak{b} \rangle = 0$$

for $\mathfrak{b} = [||\Omega||_{\omega_0} \omega_0^{n-1}]$.

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$\rightsquigarrow$ use this as an obstruction!

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Corollary

Suppose $\mathfrak{s} \in \mathfrak{S}$ *and* $\mathfrak{b} \in H^{n-1,n-1}_{BC}(X, \mathbb{R})$ *be such that*

$$\langle \mathcal{F}^{\mathfrak{s}}, \mathfrak{b} \rangle \neq 0$$

then, there exists no solution (ω, h_0, h_1) *solution to the Hull-Strominger system in* $Q_{\mathfrak{s}}$ *with* $[||\Omega||_{\omega}, \omega^{n-1}] = \mathfrak{b}$.

Futaki invariants for the Hull-Strominger system

Corollary

Let (X, Ω) Calabi-Yau and V_0, V_1 holomorphic vector bundles. Suppose $\mathfrak{b}$ balanced class such that

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Then, there is no solution to the Hull-Strominger system (ω, h_0, h_1) with $[||\Omega||_\omega \omega^2] = \mathfrak{b}$.

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Fact:

- If X satisfies the $\partial\bar{\partial}$-lemma, then $\mathfrak{S} = \{*\} \rightsquigarrow \{\mathcal{F}^{\mathfrak{s}}\}$ reduces to $\mathcal{F}_0$.

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Fact:

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$\rightsquigarrow$ may have salient implications in the geometrization of Reid's fantasy // string landscape.

Proposition (Garcia-Fernandez, GM)

Let $Q_{\mathfrak{s}}$ be the hol. Courant algebroid associated to (ω, h), let $\mathfrak{b}_0 = [\omega_0^{n-1}]$ be a balanced class and let $\varphi = (\phi, \sigma_i, a_i, b) \in H^0(X, \mathrm{End}\, Q_{\mathfrak{s}})$. Then:

$$\begin{aligned}\langle \mathcal{F}^{\mathfrak{s}}(\varphi), \mathfrak{b}_0\rangle = &-\int_X (\mathrm{tr}_{g,g_0} R_{\nabla^B} + \langle \Lambda_{\omega_0} F_h, F_h\rangle)(e_k, \phi^{*_g} e_k^{0,1} - \phi e_k^{1,0}) \frac{\omega_0^n}{n} - \int_X \langle \mathrm{tr}_{g,g_0} R^{0,2}_{\nabla^B}, b\rangle_g \frac{\omega_0^n}{n} \\ &+ \int_X \Big(\mathrm{tr}_{\mathrm{End}\, V_i}(\sigma[\Lambda_{\omega_0} F_h, \cdot]) + \langle \sigma F_h(e_j^0, e_k), F_h(\mathcal{J} e_j^0, e_k)\rangle\Big) \frac{\omega_0^n}{n} \\ &+ 2\int_X \langle \alpha(e_k), \Lambda_{\omega_0} \nabla^{h,-}_{e_k} F_h + F_h(\mathcal{J} e_j^0, g^{-1} d^c \omega(e_j^0, e_k, \cdot))\rangle \frac{\omega_0^n}{n}\end{aligned}$$

for any choices of g-orthonormal basis $e_1, \ldots, e_{2n}$ and g_0-orthonormal basis $e_1^0, \ldots, e_{2n}^0$.

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$$dd^c\omega_0 - \alpha\text{tr } F_{h_0} \wedge F_{h_0} = 0,\ \alpha = (3 + b^2)/\pi(m^2 + n^2 + p^2) > 0 \rightsquigarrow Q_0$$

Futaki invariants for the Hull-Strominger system

Proposition (Garcia-Fernandez, GM)

Let $(X_{b,\Gamma}, \Omega)$ as above. Let $\mathfrak{S}$ be the classyfing space of hol. Courant algebroids extending $\mathcal{L}$. Then,

- $\dim \mathfrak{S} = 4$
- For any $\mathfrak{s} \in \mathfrak{S}$, $\mathfrak{s} \neq 0$:

$$\langle \mathcal{F}^{\mathfrak{s}}, \mathfrak{b}_0 \rangle \neq 0,$$

In particular, any $Q_{\mathfrak{s}}$ with $\mathfrak{s} \neq 0$ does not support a solution to the Hull-Strominger system (ω, h) such that $[||\Omega||_{\omega}\omega^2] = \mathfrak{b}_0$.

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Remark: For any hol. Courant algebroid $Q_{\mathfrak{s}}$, we either find an obstruction or find a solution to Hull-Strominger with respect to $\mathfrak{b}_0$.

Higgs fields, harmonic metrics and GIT stability

$(Q, D^{\mathbf{G}})$ from a solution to Hull-Strominger.

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Recall: $\mathbf{G}$ is not positive unless $\operatorname{rk} V_0 = 0 \rightsquigarrow$ no slope stability.

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$\rightsquigarrow$ need a different GIT stability notion.

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$$\nabla^{\mathbf{H}}(J\Psi) \wedge \omega^{n-1} = 0$$

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Question: Does $(Q, D^{\mathbf{G}})$ carry a harmonic metric? $\rightsquigarrow$ fulfilled in Examples.

If so, can propose a sensible notion of GIT stability:

Higgs fields, harmonic metrics and GIT stability

Definition

Let $(Q, \langle\cdot,\cdot\rangle, D)$ be an orthogonal holomorphic vector bundle and suppose

$$D^{0,1} = \overline{\partial}_Q \quad , \quad F_D \wedge \omega^{n-1} = 0$$

Q is $[\omega^{n-1}]$-semistable if for any isotropic coherent subsheaf $\mathcal{F}$ that is D-preserved

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Future directions

- Evaluate $\mathcal{F}_0$ on algebraic Calabi-Yau threefolds
- Answer:

  Question: Does $(Q, D^{\mathbf{G}})$ carry a harmonic metric?
- Hitchin-Kobayashi correspondence? At least: produce GIT obstructions.
- Interaction between moduli structures.

Thank you for your attention!