Generalized Geometry in Interaction

ICMAT, Madrid, 14/06/2022

Moduli of Connections on Gerbes and Courant Algebroids

Severin Bunk

joint work with Carlos A. Shahbazi

UNIVERSITY OF OXFORD

Introduction and motivation

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Goal: Understand how these perspectives **differ or agree**, with an eye towards moduli spaces of SuGra solutions.

Warm-up

Connections on principal bundles and Atiyah algebroids

Principal bundles, locally

Let M a manifold with differentiably good open covering $\mathcal{U} = \{U_a\}_{a \in \Lambda}$.

- Let G be a Lie group. A **principal G-bundle on M** can be given by:

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Remark: One can enhance all these constructions to encode **smooth families** of bundles, connections and morphisms on $\mathcal{U}$.

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- Assigning to a bundle its **set** of connections gives us a **functor**

$$\mathrm{Con}\colon \mathcal{B}\mathrm{un}(\mathcal{U};G) \longrightarrow \mathcal{S}\mathrm{et}\,.$$

In particular, this **encodes the action of $\mathrm{Gau}(P)$ on $\mathrm{Con}(P)$**.

Atiyah algebroids

Let $P \in \mathcal{B}\mathrm{un}(\mathcal{U}; G)$.

- The **Atiyah algebroid of P** is the vector bundle $\mathrm{At}(P)$ defined as

$$(X + \xi_a) \in TU_a \oplus \mathfrak{g}\,, \quad (X + \xi_a) \sim (X, g_{ab}^{-1} \xi_a g_{ab} + \iota_X g_{ab}^{-1} \mathrm{d} g_{ab})$$

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$$\big[[X + \xi_a], [Y + \eta_a]\big] = \big[[X, Y]_{\mathfrak{X}} + [\xi_a, \eta_a]_{\mathfrak{g}} + \pounds_X \eta_a - \pounds_Y \xi_a\big]$$

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- There is a short exact sequence (SES) of vector bundles

$$0 \longrightarrow \mathrm{Ad}(P) \longrightarrow \mathrm{At}(P) \xrightarrow{\pi_P} TM \longrightarrow 0$$

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$\Longrightarrow$ We can **equivalently** study the moduli space/stack of connections on P through:

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Thus, the **moduli space of connections on** P is

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Note that $\mathrm{Gau}(P) \neq \mathsf{Aut}(\mathrm{At}(P))$!!

Geometric models for B-fields I

The (higher) principal bundle perspective

Gerbes

Various geometric models: certain sheaves of groupoids [Giraud; Brylinski], bundle gerbes [Murray], maps of higher presheaves [Schreiber et al.], ...

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- A **gerbe on M** (subordinate to $\mathcal{U}$) is given by a Čech 2-cocycle

$$\mathcal{G} = \{g_{abc} \colon U_{abc} \to \mathrm{U}(1)\}_{a,b,c\in\Lambda}\,, \quad g_{abc}\, g_{bcd} = g_{abd}\, g_{acd} \;\Leftrightarrow\; \delta g = 1$$

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- **Extra level:** A **2-isomorphism** $\begin{tikzcd} \mathcal{G} \arrow[r, bend left=50, "h'"{name=U}] \arrow[r, bend right=50, "h"'{name=D}] & \mathcal{G}' \arrow[from=D, to=U, Rightarrow, "\psi"] \end{tikzcd}$ is given by

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- We obtain a (symmetric monoidal) **2-groupoid** $\mathcal{G}\mathrm{rb}(\mathcal{U})$.
- This has $\pi_0\big(\mathcal{G}\mathrm{rb}(\mathcal{U}), \otimes\big) \cong \mathrm{H}^3(M;\mathbb{Z})$ [Brylinski; Murray, Stevenson; Waldorf].

Gerbes with connection

- A **gerbe with connection** is a triple $(\mathcal{G}, A, B)$, where $\mathcal{G}$ is a gerbe,

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- A **morphism** $(\mathcal{G}, A, B) \to (\mathcal{G}', A', B')$ is a pair (h, C)—a 'twisted bundle with connection'—with $h\colon \mathcal{G} \to \mathcal{G}'$ a morphism of gerbes and

$$C = \left\{C_a \in \Omega^1(U_a;\mathbb{R})\right\}_{a\in\Lambda}\,, \quad \mathrm{d}C_a = B'_a - B_a\,,$$
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- A **2-isomorphism** $\psi\colon (h, C) \to (h', C')$ is a 2-iso $\psi\colon h \to h'$ with

$$C_a = C'_a + \mathrm{d}\log\psi_a$$

We obtain a **2-groupoid** $\mathcal{G}\mathrm{rb}_\nabla(\mathcal{U})$ with a functor $\mathcal{G}\mathrm{rb}_\nabla(\mathcal{U}) \longrightarrow \mathcal{G}\mathrm{rb}(\mathcal{U})$.

Relevance: Strings/SuGra, 2-bundles [R. Téllez' talk], twisted bundles and K-theory, functorial field theories, ...

The groupoid of connections on a gerbe

Definition [SB, Shahbazi]

Define $\mathrm{Con}_\delta(\mathcal{G})$ as the fibre of $\mathcal{G}\mathrm{rb}_\nabla(\mathcal{U}) \to \mathcal{G}\mathrm{rb}(\mathcal{U})$ over $\mathcal{G}$.

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Example: $\mathcal{G} = \mathcal{I}$ the trivial gerbe, i.e. $g \equiv 1$. Then,

$$\pi_0\big(\mathrm{Con}_\delta(\mathcal{I}), \otimes\big) \cong \frac{\Omega^2(M)}{\mathrm{d}\Omega^1(M)}, \quad \pi_0\big(\mathrm{Con}_{\flat;\delta}(\mathcal{I}), \otimes\big) \cong \mathrm{H}^2_{\mathrm{dR}}(M),$$

$$\pi_1\big(\mathrm{Con}_\delta(\mathcal{I}), \otimes\big) \cong \pi_1\big(\mathrm{Con}_{\delta,\flat}(\mathcal{I}), \otimes\big) \cong \Omega^1_{cl}(M) \neq 0 .$$

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- This acts on $\mathrm{Con}_\delta(\mathcal{G})$ and $\mathrm{Con}_{\flat;\delta}(\mathcal{G})$ (e.g. as simplicial groups).
- The associated **higher action groupoids**, i.e. the **derived/homotopy quotients** $\mathrm{Mdl}_\delta(\mathcal{G}) = \mathrm{Con}_\delta(\mathcal{G}) /\!\!/ \mathrm{Gau}_\delta(\mathcal{G})$ satisfy

(Sl. 13) .

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Geometric models for B-fields II

The (higher) Atiyah algebroid perspective

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- To $(\mathcal{G}, A) \in \mathcal{G}\mathrm{rb}_a(\mathcal{U})$, we associate an **exact Courant algebroid** $E(\mathcal{G}, A)$; this is glued together as [Hitchin]

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Proposition [SB, Shahbazi]

This extends to a functor $E\colon \mathrm{h}\mathcal{G}\mathrm{rb}_a^{(n.i.)}(\mathcal{U}) \longrightarrow \mathrm{ECAlgd}(M)$.

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Homotopy quotients of connections

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We have **two candidate notions of B-fields**: $\mathrm{Con}_\delta(\mathcal{G})$ and $\mathrm{Con}_\delta(\mathcal{G}, A)$, resulting in **higher groupoids** $\mathrm{Mdl}_\delta(\mathcal{G})$ and $\mathrm{Mdl}_\delta(\mathcal{G}, A)$ **of B-fields**.

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Theorem [SB, Shahbazi]

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In this sense, **the two models for B-fields are equivalent!**

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$\Rightarrow$ The above two ∞-colimits are **canonically equivalent** in $\mathcal{G}\mathrm{rpd}_\infty$. $\square$

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- We thus obtain **homotopy quotients** for each $k = 0, \dots, n$,

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Generalisations I: Higher gerbes

- An n-**gerbe** on M is a Čech $(n+1)$-cocycle g valued in $\mathrm{U}(1)$.
- A **connection on an n-gerbe** consists of $(A_1, \ldots, A_{n+1})$, with A_k certain families of k-forms on $(n+2-k)$-fold intersections of patches.
- We have $\pi_0\big(\mathcal{G}\mathrm{rb}^n_\nabla(\mathcal{U})\big) \cong \hat{\mathrm{H}}^{n+1}(M;\mathbb{Z})$.
- Write $\mathcal{A}^{(k)} = (A_1, \ldots, A_k)$ for a **partial connection** on $\mathcal{G}$.

  An n-gerbe carries n **levels** of partial connection;

$$\mathcal{G}\mathrm{rb}^n_\nabla(\mathcal{U}) \xrightarrow{p^{n+1}_n} \mathcal{G}\mathrm{rb}^n_{\nabla|n}(\mathcal{U}) \xrightarrow{p^n_{n-1}} \cdots \xrightarrow{p^n_{n-1}} \mathcal{G}\mathrm{rb}^n_{\nabla|1}(\mathcal{U}) \xrightarrow{p^1_0} \mathcal{G}\mathrm{rb}^n(\mathcal{U})\,.$$

- We thus obtain **homotopy quotients** for each $k = 0, \ldots, n$,

$$\mathrm{Mdl}_\delta(\mathcal{G}, \mathcal{A}^{(k)}) = \mathrm{Con}_\delta(\mathcal{G}, \mathcal{A}^{(k)})/\!\!/\mathrm{Gau}_\delta(\mathcal{G}, \mathcal{A}^{(k)})\,.$$

Remark: We get rid of the dependence on $\mathcal{U}$ by taking the colimit over all good coverings and their refinements.

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Set $\mathcal{H} = \mathcal{F}\mathrm{un}(\mathcal{C}\mathrm{art}^{\mathrm{op}}, \mathcal{G}\mathrm{rpd}_\infty)$; view its objects as **smooth spaces**.

- Let $\mathcal{G}$ be an n-gerbe with partial connection $\mathcal{A}^{(k)}$.
- For $0 \leqslant l \leqslant k \leqslant n$, let $\mathcal{A}^{(l)}$ be the partial connection obtained from $\mathcal{A}^{(k)}$ by forgetting part of the data.

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Theorem [SB, Shahbazi]

There are **canonical equivalences**

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Remark: A similar theorem holds true with gauge transformations replaced by **automorphisms which cover (certain) diffeomorphisms** of the base manifold M. [SB, Shahbazi]

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- Applications to **SuGra in non-trivial topology**? Since gerbes and B-fields are sensitive to this.
- ...

Thank you for your attention!