Brane quantization of toric Poisson varieties

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Motivation: Categories of generalized boundary conditions

A generalized complex (GC) structure is a complex structure on $TM \oplus T^*M$:

$$\mathbb{I} : TM \oplus T^*M \to TM \oplus T^*M, \qquad \mathbb{I}^2 = -1.$$

The integrability condition is with respect to the *Courant bracket*.

Motivation: Categories of generalized boundary conditions

Examples:

- Symplectic structure $\omega : TM \to T^*M$

$$\mathbb{I}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

- Complex structure $I : TM \to TM$

$$\mathbb{I}_I = \begin{pmatrix} I & 0 \\ 0 & -I^* \end{pmatrix}.$$

Motivation: Categories of generalized boundary conditions

We would like to define categories of boundary conditions for GC structures.
Examples:

- Symplectic structure ω: A-model

$$\mathcal{C}_\omega = \mathit{Fuk}(\omega).$$

  Objects: Lagrangians with local systems, Coisotropic branes (Kapustin-Orlov).

- Complex structure I : B-model

$$\mathcal{C}_I = \mathit{Coh}(I),$$

  Objects: Complex submanifolds with holomorphic vector bundles.

What about other GC structures?

Motivation: Categories of generalized boundary conditions

The deformations of the A and B-models match the deformations of the corresponding GC structures. For example, the deformations of the B-model are given by

$$H^0(\wedge^2 T) \oplus H^1(T) \oplus H^2(\mathcal{O}).$$

- $H^1(T)$: infinitesimal deformations of complex structure.
- $H^0(\wedge^2 T) \oplus H^2(\mathcal{O})$ take you out of complex into generalized complex.

We're interested mostly in $H^0(\wedge^2 T)$ which corresponds to the 'non-commutative' deformations of $Coh(I)$.

Holomorphic Poisson deformation

- Holomorphic Poisson structure: $\sigma \in H^0(\wedge^2 T_M)$, such that

$$\overline{\partial}\sigma = 0, \qquad [\sigma, \sigma] = 0.$$

- $\sigma = -\frac{1}{4}(P + iQ)$
- Deformation of $\mathbb{I}_I$:

$$\mathbb{I}_{\hbar\sigma} = \begin{pmatrix} I & \hbar Q \\ 0 & -I^* \end{pmatrix}.$$

The category of branes $\mathcal{C}_{\hbar\sigma}$ should give a 'non-commutative' deformation of $Coh(I)$. This would give a *geometric approach* to non-commutative geometry. Let's ignore the difficulties and see what we can do.

Geometric quantization of Kähler manifolds: B-model

1. Compact symplectic manifold: (M, ω),
2. Prequantization: (L, ∇) such that

$$\nabla^2 = -2\pi i\omega.$$

3. Complex polarization: complex structure I such that

$$\omega I + I^*\omega = 0, \qquad I^*\omega > 0.$$

4. Quantization is given by the global holomorphic sections:

$$H^0(M, L) = \ker \nabla^{0,1}.$$

Homogeneous coordinate ring: B-model

- The complex structure I plays an auxiliary role: it tells us which subset of sections of L to select, but it is not encoded in the quantization. In good cases, the quantization is invariant under deformations of I.

Homogeneous coordinate ring: B-model

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- To encode the complex structure, we must quantize all integral multiples: $\omega, 2\omega, 3\omega, \ldots$ to obtain the homogeneous coordinate ring

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- Kodaira embedding theorem: This algebra encodes (M, I).
- If we deform the polarization I to a 'generalized complex polarization' $\mathbb{I}_{\hbar\sigma}$ by turning on a Poisson structure, the product on $\mathcal{A}$ will deform to give a quantization of σ.

Homogeneous coordinate ring: B-model

Recast $\mathcal{A}$ purely in terms of the category $Coh(I)$:

$$\mathcal{A} = \bigoplus_{n\geq 0} H^0(M, L^{\otimes n}) = \bigoplus_{n\geq 0} Hom_I(\mathcal{O}, \Phi^n(\mathcal{O})),$$

where

- $\mathcal{O} \in Coh(I)$ is the trivial line bundle,
- Φ is an autoequivalence of $Coh(I)$:

$$\Phi : \mathcal{E} \mapsto \mathcal{E} \otimes L.$$

When we turn on σ, the category $Coh(I)$ should deform to $\mathcal{C}_{\hbar\sigma}$. We must also deform $\mathcal{O}$ and Φ along with it.

Branes on Poisson varieties

The objects of the conjectural category $\mathcal{C}_{\hbar\sigma}$ were defined by Gualtieri.

Definition

A brane on $\mathbb{I}_\sigma$ is a unitary line bundle (L, ∇) such that

$$FI + I^*F + FQF = 0,$$

where $\nabla^2 = -2\pi i F$.

There is always a trivial solution given by $(\mathcal{O}, \nabla = d)$.

Branes on Poisson varieties

The deformation of Φ was also given in the work of Gualtieri.

Theorem (Gualtieri)

(L, ∇) a holomorphic prequantization of ω. If L admits a Poisson module structure, then the functor $\Phi = L \otimes (-)$ deforms to an autoequivalence T of $\mathcal{C}_\sigma$.

Starting with the brane $(\mathcal{O}, d)$, we get a sequence of branes $T^k(\mathcal{O})$.

Branes on Poisson varieties

Constructing the equivalence T:

- Let h be the metric on L, and let ∇ be the Chern connection.
- Let $\hat{Q}$ be the Poisson module, which is a lift of Q to $tot(L)$.
- Let $W = \hat{Q}(d \log h)$, a Poisson vector field on M, with flow φ_t.
- Then $T(B) = (L, \overline{\nabla}) \otimes \varphi_1^* B$, where

$$\overline{\nabla} = \int_0^1 \varphi_t^*(\nabla) dt.$$

Non-commutative homogeneous coordinate ring

Having obtained deformations $\mathcal{O}$ and T, we get the deformed homogeneous coordinate ring:

$$\mathcal{A}_\hbar = \bigoplus_{n \geq 0} Hom_{\mathcal{C}_{\hbar\sigma}}(\mathcal{O}, T^n(\mathcal{O})).$$

Major Problem: We don't know how to compute the homomorphisms in $\mathcal{C}_{\hbar\sigma}$. We're stuck!

In this project we develop a new approach to the category of boundary conditions which uses the A-model of an auxiliary symplectic manifold. We implement this for the deformation quantization of toric Poisson varieties.

Brane quantization of toric Poisson varieties

Francis Bischoff* Marco Gualtieri†

Abstract

In this paper we propose a noncommutative generalization of the relationship between compact Kähler manifolds and complex projective algebraic varieties. Beginning with a prequantized Kähler structure, we use a holomorphic Poisson tensor to deform the underlying complex structure into a generalized complex structure, such that the prequantum line bundle and its tensor powers deform to a sequence of generalized complex branes. Taking homomorphisms between the resulting branes, we obtain a noncommutative deformation of the homogeneous coordinate ring. As a proof of concept, this is implemented for all compact toric Kähler manifolds equipped with an R-matrix holomorphic Poisson structure, resulting in what could be called noncommutative toric varieties.

To define the homomorphisms between generalized complex branes, we propose a method which involves lifting each pair of generalized complex branes to a single coisotropic A-brane in the real symplectic groupoid of the underlying Poisson structure, and compute morphisms in the A-model between the Lagrangian identity bisection and the lifted coisotropic brane. This is done with the use of a multiplicative holomorphic Lagrangian polarization of the groupoid.

Contents

Geometric quantization: A-model approach

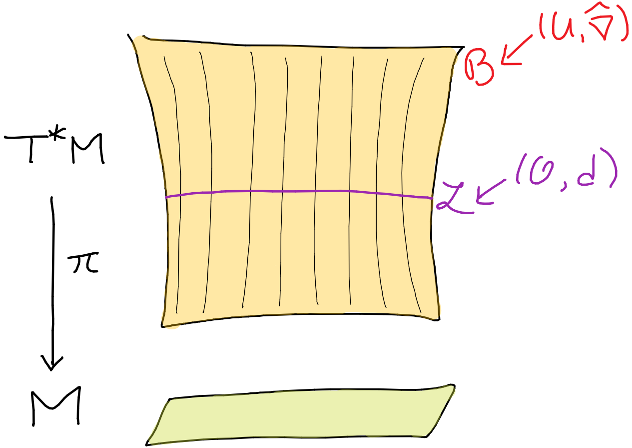
Encode Kähler geometry $(M, I, \omega, (L, \nabla))$ as a pair of A-branes in $(T^*M, \omega_0 = Im(\Omega_0))$.

1. Lagrangian brane: $\mathcal{L} = (0_M, \mathcal{O}, d)$ (zero section)
2. Space-filling brane: $\mathcal{B} = (U, \hat{\nabla}) = (\pi^* L, \pi^* \nabla - 2\pi i Re(\alpha_0))$, where $d\alpha_0 = \Omega_0$, the canonical holomorphic symplectic form.
   - $F = \frac{i}{2\pi} \hat{\nabla}^2$.
   - $F + i\omega_0 = \Omega_0 + \pi^* \omega$, the twisted cotangent bundle.

Definition. $(U, \hat{\nabla})$ is a space-filling brane in (T^*M, ω_0) if $F + i\omega_0$ is a holomorphic symplectic form.

Intersection recovers Kähler structure: $(U, \hat{\nabla})|_{\mathcal{L}} = (L, \nabla)$.

Geometric quantization: A-model approach



Geometric quantization: A-model approach

The quantization is defined as the space of homomorphisms in the A-model between the branes (Gukov-Witten):

$$Hom_{\omega_0}(\mathcal{L}, \mathcal{B}).$$

Proposal to compute this based on **holomorphic geometric quantization**:

- Holomorphic prequantization of $\mathcal{B}$: $(U, D) = (\pi^* L, \pi^* \nabla - 2\pi i \alpha_0)$.
- Holomorphic polarization $\mathcal{F}$ of $F + i\omega_0$: Fibres of projection

$$\pi : T^* M \to M.$$

- $Hom_{\omega_0}(\mathcal{L}, \mathcal{B})$ defined to be holomorphic D-flat sections along leaves intersecting $\mathcal{L}$:

$$Hom_{\omega_0}(\mathcal{L}, \mathcal{B}) \cong H^0(M, L) = Hom_I(\mathcal{O}, L).$$

Homogeneous coordinate ring: A-model approach

The cotangent bundle (T^*M, ω_0) is a symplectic groupoid:

$$\oplus : T^*M \times_M T^*M \to T^*M.$$

This endows the A-model with a tensor product.

- The zero section is a monoidal unit: $\mathcal{L} * \mathcal{L} = \mathcal{L}$.
- The rescalings $k\omega$ are encoded by the tensor powers $\mathcal{B}^{*k}$.

Homogeneous coordinate ring given by

$$\mathcal{A} = \bigoplus_{n \geq 0} Hom_{\omega_0}(\mathcal{L}, \mathcal{B}^{*n}),$$

with product induced by tensor product of branes.

Generalized complex branes: A-model approach

Given GC branes B_1 and B_2 for a GC structure $\mathbb{I}$, how can we compute the space of homomorphisms $Hom_{\mathbb{I}}(B_1, B_2)$?

- A GC structure $\mathbb{I}$ has an underlying real Poisson structure Q:

$$\mathbb{I} = \begin{pmatrix} * & Q \\ * & * \end{pmatrix}.$$

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- We make use of the A-model of this symplectic manifold. It has a tensor structure which comes from the multiplication on $\mathcal{G}$. The unit object is the *identity bisection* $\mathcal{L}$.

Generalized complex branes: A-model approach

- The map $(t, s) : (\mathcal{G}, \omega_0) \to (M \times M, \mathbb{I} \times \mathbb{I}^T)$ is *generalized holomorphic* (Crainic).

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- Given GC branes B_i, B_j on $\mathbb{I}$, we obtain an A-brane in $(\mathcal{G}, \omega_0)$:

$$\mathcal{B}_{i,j} = (t,s)^{-1}(B_i \times B_j^*).$$

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- We then define the space of homomorphisms using the A-model:

$$Hom_{\mathbb{I}}(B_j, B_i) := Hom_{\omega_0}(\mathcal{L}, \mathcal{B}_{i,j}),$$

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- The composition of morphisms is induced by the tensor product.

Implementation: Toric Poisson structures

- Toric Kähler structure: $(S^1)^n \subset (\mathbb{C}^\times)^n \curvearrowright_\rho (M, \omega, I)$.
- Lift to holomorphic Hamiltonian action $(\mathbb{C}^\times)^n \curvearrowright (T^*M, \Omega_0)$ with moment map
$$J_0 : T^*M \to \mathbb{C}^n.$$
- Equivariant prequantization (L, ∇), $\hat{\rho} : (\mathbb{C}^\times)^n \to Aut(L)$.
- R-matrix $C \in \wedge^2\mathbb{R}^n$. This defines a holomorphic Poisson structure on M:
$$\sigma_C = \rho_*(C),$$
and Poisson module structure on L:
$$\hat{\sigma}_C = \hat{\rho}_*(C).$$

Implementation: Toric Poisson structures

The Poisson structure σ_C deforms I into GC structure:

$$\mathbb{I}_{\hbar\sigma_C} = \begin{pmatrix} I & \hbar Q \\ 0 & -I^* \end{pmatrix}.$$

And the Poisson module structure $\hat{\sigma}_C$ allows us to deform $\Phi = L \otimes (-)$ into an autoequivalence T of $\mathcal{C}_{\hbar\sigma_C}$.

Implementation: Toric Poisson structures

Symplectic groupoid of σ_C is a deformation of T^*M:

- The underlying symplectic manifold is given by T^*M,
- Structure maps deformed: target and source

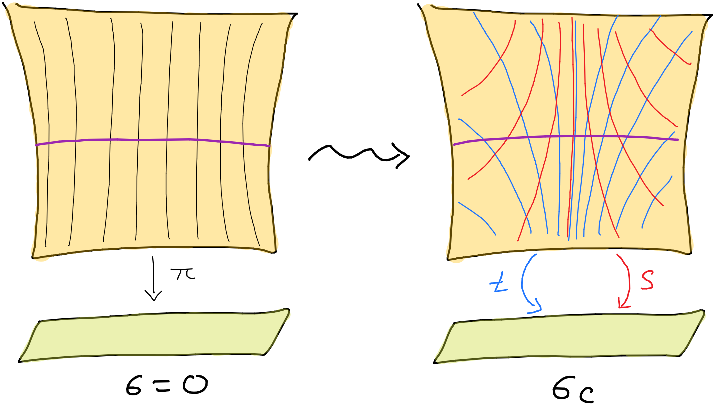
$$t(z) = e^{\frac{1}{2}CJ_0(z)}\pi(z), \qquad s(z) = e^{-\frac{1}{2}CJ_0(z)}\pi(z),$$

  multiplication

$$m(x, y) = e^{-\frac{1}{2}CJ_0(y)}x + e^{\frac{1}{2}CJ_0(x)}y.$$

As a result, the A-model is unchanged, but its tensor structure is deformed.

Implementation: Toric Poisson structures



Implementation: Toric Poisson structures

In order to construct the deformed homogeneous coordinate ring:

- We construct the family of branes $B_n = T^n(\mathcal{O}) = (L^{\otimes n}, \overline{\nabla^{\otimes n}})$, where

$$\overline{\nabla^{\otimes n}} = \frac{1}{n}\int_0^n (\varphi_t^* \nabla^{\otimes n}) dt.$$

- We pullback to get branes on T^*M:

$$\mathcal{B}_{n,0} = (t,s)^{-1}(B_n \times \mathcal{O}).$$

- **Theorem (B, Gualtieri)** $\mathcal{B}_{n,0} \cong \mathcal{B}^{*n}$.
- The algebra:

$$\mathcal{A}_C = \bigoplus_{n\geq 0} Hom_{\omega_0}(\mathcal{L}, \mathcal{B}^{*n}).$$

  As a vector space, this does not change, but the product is deformed via the deformed tensor product.

Twisted graded monoid

What geometric structures underly the collection of branes $\mathcal{B}_{n,0}$?

1. The brane $\mathcal{B}_{n,0}$ equips T^*M with the structure of a prequantized holomorphic symplectic manifold: $(Z_n, \Omega_n, U_n, D_n)$. In fact, these spaces are holomorphic symplectic Morita equivalences of σ_C.
2. The tensor product structure for the branes translates to the composition of holomorphic symplectic Morita equivalences:
$$m : Z_k \times_M Z_r \to Z_{k+r}.$$
3. These lift to the prequantizations:
$$\Theta : (U_k, D_k) \boxtimes (U_r, D_r)|_{Z_k \times_M Z_r} \to (U_{k+r}, D_{k+r}).$$
This is given by
$$\Theta_{(x,y)}(f, g) = e^{i\pi C(J(x),J(y))}(e^{-\frac{1}{2}CJ(y)} f) \otimes (e^{\frac{1}{2}CJ(x)} g).$$

Twisted graded monoid

We summarise this by saying that

$$\sqcup_{n\in\mathbb{Z}}(Z_n, \Omega_n, U_n, D_n)$$

is a holomorphic symplectic groupoid equipped with multiplicative prequantization.

Quantization

The quantization $Hom_{\omega_0}(\mathcal{L}, \mathcal{B}_{1,0})$ is computed as before, via holomorphic geometric quantization.

- We use the same holomorphic prequantization of $\mathcal{B}_{1,0}$: (U_1, D_1).
- The polarization via fibres is no longer good, since it does not respect the multiplication.
- Instead we take as holomorphic polarization $\mathcal{F}$ the fibres of holomorphic moment map

$$J : (Z_1, \Omega_1) \to \mathbb{C}^n,$$

  which is constructed from J_0 and the moment map for ω.
- Quantization

$$Hom_{\omega_0}(\mathcal{L}, \mathcal{B}_{0,1}) = \bigoplus_{\text{BS leaves intersecting } \mathcal{L}} H^0(U_1)^{D_1|\mathcal{F}}.$$

Quantization

The polarization $\mathcal{F}$ behaves differently:

1. Not all fibres admit holomorphic flat sections:

$$\text{BS Leaves} \iff \text{Lattice points } \mathbb{Z}^n \subset \mathbb{C}^n.$$

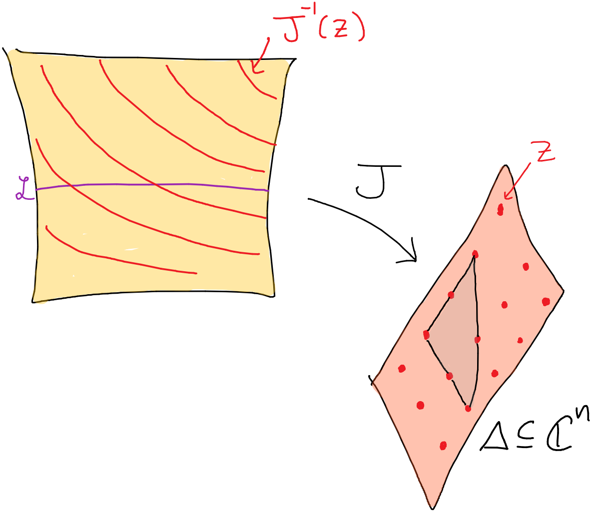
2. Not all fibres intersect $\mathcal{L}$:

$$J^{-1}(z) \cap \mathcal{L} \neq \emptyset \iff z \in \Delta$$

Therefore, we obtain the weight space decomposition of $H^0(M, L)$:

$$Hom_{\omega_0}(\mathcal{L}, \mathcal{B}_{0,1}) = \bigoplus_{z \in \Delta \cap \mathbb{Z}^n} \mathbb{C}z.$$

Quantization



Non-commutative homogeneous coordinate ring

Theorem (B, Gualtieri)

The algebra $\mathcal{A}_C$ and the homogeneous coordinate ring of the toric variety have canonically isomorphic underlying graded vector spaces

$$\mathcal{A}_C \cong \bigoplus_{n \geq 0} H^0(M, L^{\otimes n}).$$

Under this identification the product of two homogeneous sections f and g, with respective $\mathbb{T}_{\mathbb{C}}$-weights $w_1, w_2 \in \mathfrak{t}^*_{\mathbb{C}}$, is given by

$$f * g = e^{\frac{i}{4\pi} C(w_1, w_2)} f \otimes g,$$

resulting in the commutation relations

$$f * g = e^{\frac{i}{2\pi} C(w_1, w_2)} g * f.$$

Thank You