Embedding superconformal vertex algebras from Killing spinors and (0,2) mirror symmetry

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Subject of this talk

Interaction between

- **Superconformal algebras**
- **Super current algebras**
- **Quadratic Lie algebras**
- **Courant algebroids**

The Virasoro algebra

Punctured disc: $\mathring{\mathbb{D}}^1 = \mathrm{Spec}\,\mathbb{C}[t^{\pm 1}]$
Lie algebra of vector fields:

$$\mathrm{Vect}(\mathring{\mathbb{D}}^1) = \mathrm{Der}_{\mathbb{C}}(\mathbb{C}[t^{\pm 1}]) = \mathbb{C}[t^{\pm 1}]\frac{\mathrm{d}}{\mathrm{d}t}$$

Nice basis of vector fields:

$$\mathtt{L}_n := -t^{n+1}\frac{\mathrm{d}}{\mathrm{d}t}, \qquad \text{for } n \in \mathbb{Z}$$

Lie bracket: $[\mathtt{L}_m, \mathtt{L}_n] = (m-n)\mathtt{L}_{m+n}$

Virasoro algebra: the central extension

$$0 \to \mathbb{C}\mathtt{C} \longrightarrow \mathtt{Vir} \longrightarrow \mathrm{Vect}(\mathring{\mathbb{D}}) \to 0,$$

i.e., $\mathtt{Vir} = \mathrm{Vect}(\mathring{\mathbb{D}}) \oplus \mathbb{C}\mathtt{C}$ as a vector space, with commutators

$$[\mathtt{L}_m, \mathtt{L}_n] = (m-n)\mathtt{L}_{m+n} + \delta_{m+n,0}(m^3 - m)\frac{\mathtt{C}}{12},$$

$$[\mathtt{C}, -] = 0.$$

Define $\mathrm{V}^c(\mathtt{Vir}) = \mathrm{U}(\mathtt{Vir})/(\mathtt{C} - c1)$ (localize the universal enveloping algebra at central charge $c \in \mathbb{C}$).

Affinization of quadratic Lie algebras

Finite-dimensional quadratic Lie $\mathbb{C}$-algebra: $(\mathfrak{g}, \langle -, - \rangle)$

Loop algebra: $\mathsf{L}\mathfrak{g} = \mathsf{Map}(\mathring{\mathbb{D}}^1, \mathfrak{g}) = \mathfrak{g}[t^{\pm 1}]$ (with $\mathring{\mathbb{D}}^1 = \mathsf{Spec}\, \mathbb{C}[t^{\pm 1}]$)

Affinization or current algebra: central Lie-algebra extension

$$0 \to \mathbb{C}\mathtt{K} \longrightarrow \widehat{\mathfrak{g}} \longrightarrow \mathsf{L}\mathfrak{g} \to 0,$$

i.e., $\widehat{\mathfrak{g}} = \mathsf{L}\mathfrak{g} \oplus \mathbb{C}\mathtt{K}$ as a vector space, with commutators (for $\mathtt{a}, \mathtt{b} \in \mathfrak{g}$):

$$[\mathtt{a}t^m, \mathtt{b}t^n] = [\mathtt{a}, \mathtt{b}]t^{m+n} + m\delta_{m+n,0}\langle \mathtt{a}, \mathtt{b} \rangle \mathtt{K},$$
$$[\mathtt{K}, \mathtt{a}t^m] = 0.$$

Affinization and the Sugawara construction

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$$[\mathtt{K}, \mathtt{a}t^m] = 0.$$

Define
- $\mathrm{V}^k(\mathfrak{g}) = \mathrm{U}(\widehat{\mathfrak{g}})/(k - \mathtt{K})$ for 'level' $k \in \mathbb{C}$,
- Ω=(Casimir operator of $(\mathfrak{g}, \langle -, - \rangle) \in \mathrm{U}(\mathfrak{g})$ (UEA of $\mathfrak{g}$),
- $2h^\vee \dim \mathfrak{g} = \mathsf{Tr}(\mathsf{ad}(\Omega) \colon \mathfrak{g} \to \mathfrak{g})$ ('dual Coxeter number').

Theorem (1968)

If $k + h^\vee \neq 0$, then $\exists$ a canonical embedding $\mathrm{V}^c(\mathtt{Vir}) \subset \mathrm{V}^k(\mathfrak{g})$ of central charge $c = \frac{k \dim \mathfrak{g}}{k + h^\vee}$.

Superconformal algebras (Fattori–Kac, 2002)

Fix N odd variables $\theta^1, \ldots, \theta^N$.

Punctured superdisc: $\mathring{\mathbb{D}}^{1|N} = \mathbb{S}\mathrm{pec}\, \mathbb{C}[t^{\pm 1}, \theta^1, \ldots, \theta^N]$ (super Spec)

Define the supercontact 1-form on $\mathring{\mathbb{D}}^{1|N}$:

$$\alpha := \mathrm{d}t + \sum \theta^i \mathrm{d}\theta^i \in \Omega^1(\mathring{\mathbb{D}}^{1|N})$$

and Lie superalgebra of supercontact vector fields

$$K(1|N) := \{v \mid \exists \text{ function } f_v \text{ s.t. } L_v\alpha = f_v\alpha\} \subset \mathrm{Vect}\left(\mathring{\mathbb{D}}^{1|N}\right).$$

- For $N \leq 3$, $K(1|N)$ admits one non-trivial central extension $\mathcal{K}(1|N) = K(1|N) \oplus \mathbb{C}\mathtt{c}$ (as a Lie conformal algebra).
- For $N = 4$, there are *two* non-trivial central extensions (for the derived Lie algebra $K(1|N)' \subset K(1|N)$), i.e., two central charges.
- For $N \geq 5$, there are no non-trivial central extensions.

$N = 1$: The Neveu–Schwarz superconformal algebra

Neveu–Schwarz superconformal algebra $\mathtt{NS} = \mathcal{K}(1|1)$:
Generated by $\mathtt{L}_m, \mathtt{G}_n, \mathtt{C}$ ($m \in \mathbb{Z}, n \in \frac{1}{2} + \mathbb{Z}$), with commutation relations

$$[\mathtt{L}_m, \mathtt{L}_n] = (m-n)\mathtt{L}_{m+n} + \delta_{m+n,0}(m^3 - m)\frac{\mathtt{C}}{12},$$

$$[\mathtt{L}_m, \mathtt{G}_n] = \left(\frac{m}{2} - n\right)\mathtt{G}_{m+n}, \quad [\mathtt{C}, -] = 0,$$

$$[\mathtt{G}_m, \mathtt{G}_n] = 2\mathtt{L}_{m+n} + \delta_{m+n,0}\left(m^2 - \frac{1}{4}\right)\frac{\mathtt{C}}{3}.$$

Neveu–Schwarz and superaffinization of Lie algebras

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Finite-dimensional quadratic Lie superalgebra: $(\mathfrak{g}, \langle -, - \rangle)$

$N = 1$ super loop algebra: $\mathsf{L}^{1|1}\mathfrak{g} = \mathrm{Map}(\mathring{\mathbb{D}}^{1|1}, \mathfrak{g}) = \mathfrak{g}[t^{\pm 1}, \theta]$

Superaffinization or supercurrent algebra: $\widehat{\mathfrak{g}}_{super} = \mathsf{L}^{1|1}\mathfrak{g} \oplus \mathbb{C}\mathtt{K}$ as a vector space, with commutators (for $\mathtt{a}, \mathtt{b} \in \mathfrak{g}$):

$$[\mathtt{a}t^m, \mathtt{b}t^n] = [\mathtt{a}, \mathtt{b}]t^{m+n} + m\delta_{m+n,0}\langle \mathtt{a}, \mathtt{b}\rangle \mathtt{K}, \quad [\mathtt{K}, -] = 0,$$

$$[\mathtt{a}t^m, \mathtt{b}t^n\theta] = [\mathtt{a}, \mathtt{b}]t^{m+n}\theta, \quad [\mathtt{a}t^m\theta, \mathtt{b}t^n\theta] = \delta_{m+n,-1}\langle \mathtt{b}, \mathtt{a}\rangle \mathtt{K}.$$

The Kac–Todorov construction

Define $\mathrm{V}^c(\mathtt{NS}) = \mathrm{U}(\mathtt{NS})/(\mathtt{C} - c1)$ (localize universal enveloping algebra at central charge $c \in \mathbb{C}$)

Theorem (1985)

If $k + h^\vee \neq 0$, then $\exists$ a canonical embedding $\mathrm{V}^c(\mathtt{NS}) \subset \mathrm{V}^{k+h^\vee}(\mathfrak{g}_{super})$ of central charge $c = \frac{k \dim \mathfrak{g}}{k+h^\vee} + \frac{\dim \mathfrak{g}}{2}$.

Kac–Todorov and aims of the talk

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Algebraic problem: **obtain an $N{=}2$ SUSY version, i.e., an embedding of the $N{=}2$ superconformal algebra**

$$\mathbf{V}^c(\mathcal{K}(1|2)) \subset \mathbf{V}^k(\mathfrak{g}_{super}),$$

where now $\mathrm{V}^c(\mathcal{K}(1|2)) = \mathrm{U}(\mathcal{K}(1|2))/(\mathtt{C} - c1)$.
The most natural framework to construct any of these embedding is the theory of **vertex algebras**.

Kac–Todorov and aims of the talk

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The most natural framework to construct any of these embedding is the theory of **vertex algebras**.

Geometric problem: **apply this embedding in Borisov's vertex-algebra approach to (0,2) mirror symmetry.**

Vertex algebras: quantization on the punctured disc

Vertex algebras quantize operator-valued distributions on the punctured disc using the following principles:

- View $\mathbb{C}[z^{\pm 1}]$ as the space of **test functions**. This is a topological algebra, for the linear topology with a neighbourhood basis of $0 \in \mathbb{C}[z^{\pm}]$ given by the subspaces $z^{-N}\mathbb{C}[z] \subset \mathbb{C}[z^{\pm 1}]$ $(N \in \mathbb{Z})$.
- Interpret **quantum fields** as operator-valued 'distributions' on $\mathring{\mathbb{D}}^1$, i.e., continuous linear maps into a topological algebra $\mathcal{U}$ of operators:
$$a \colon \mathbb{C}[z^{\pm}] \longrightarrow \mathcal{U}.$$
- Find a suitable set $\mathcal{F}$ of continuous distributions that are '**local to each other**', so they can be 'multiplied'.

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- Find a suitable set $\mathcal{F}$ of continuous distributions that are '**local to each other**', so they can be 'multiplied'.

We write distributions $\mathbb{C}[z^{\pm}] \to \mathcal{U}$ as $\mathcal{U}$-valued formal Laurent series

$$a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} \in \mathcal{U}[\![z^{\pm 1}]\!]$$

($a_{(n)}$='Fourier modes') via the pairing

$$\mathbb{C}[z^{\pm 1}] \otimes \mathcal{U}[\![z^{\pm 1}]\!] \to \mathcal{U},\ \varphi \otimes a \mapsto \operatorname*{Res}_{z=0}(\varphi(z)a(z)) = \frac{1}{2\pi i}\oint \varphi(z)a(z)\mathrm{d}z.$$

Vertex algebras: locality

Suppose $\mathcal{U}$ is a Lie algebra. Two $\mathcal{U}$-valued formal distributions

$$a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}, \quad b(w) = \sum_{n\in\mathbb{Z}} b_{(n)} w^{-n-1} \in \mathcal{U}[\![z^{\pm 1}]\!],$$

are *mutually local* if $\exists p(z,w) \in \mathcal{U}[\![z,w]\!][z^{-1}, w^{-1}, (z-w)^{-1}]$ such that

$$[a(z), b(w)] = \iota_{|z|>|w|} p(z,w) - \iota_{|w|>|z|} p(z,w) \in \mathcal{U}[\![z^{\pm 1}, w^{\pm 1}]\!].$$

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In the RHS, we use canonical maps

$$\mathcal{U}[\![z,w]\!][z^{-1}, w^{-1}, (z-w)^{-1}] \xrightarrow{\boldsymbol{\iota}_{|z|>|w|}} (\mathcal{U}(\!(z)\!))(\!(w)\!)$$

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$$\mathcal{U}(\!(z)\!) = \{\mathcal{U}\text{-valued formal Laurent series } \sum_{n=-N}^{\infty} c_n z^n \text{ (with } c_n \in \mathcal{U})\}.$$

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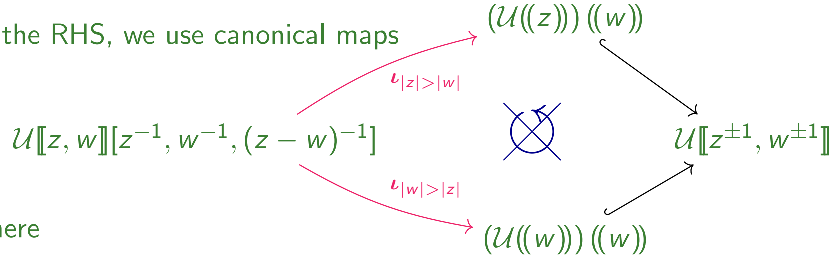

where

$$\mathcal{U}(\!(z)\!) = \{\mathcal{U}\text{-valued formal Laurent series } \sum_{n=-N}^{\infty} c_n z^n \text{ (with } c_n \in \mathcal{U})\}.$$

*: The right-hand side measures the failure of the diagram to commute. Then $[a(z), b(w)]$ can be multiplied by $\mathbb{C}(\!(z)\!)$ and $\mathbb{C}(\!(w)\!)$.

Vertex algebras: Operator Product Expansion

Two $\mathcal{U}$-valued formal distributions

$$a(z) = \sum_{m\in\mathbb{Z}} a_{(m)} z^{-m-1} \in \mathcal{U}[\![z^{\pm 1}]\!], \quad b(w) = \sum_{n\in\mathbb{Z}} b_{(n)} w^{-n-1} \in \mathcal{U}[\![w^{\pm 1}]\!],$$

are local to each other iff $\exists$ sequence $\{c^j(w)\}_{j=1}^N \subset \mathcal{U}[\![w^{\pm 1}]\!]$ such that

$$[a(z), b(w)] \stackrel{\text{def}}{=} \sum_{m,n\in\mathbb{Z}} [a_{(m)}, b_{(n)}] z^{-m-1} w^{-n-1} = \sum_{j=0}^{N} \frac{1}{j!} \partial_w^j \delta(z-w)\, c^j(w).$$

In this case, one usually writes an **OPE**

$$a(z)b(w) \sim \sum_{j=0}^{N} \frac{c^j(w)}{(z-w)^{j+1}}$$

and defines the **$\boldsymbol{\lambda}$-bracket**

$$[a_\lambda b] = \sum_{j=0}^{N} \frac{\lambda^j}{j!} c^j \in \left(\mathcal{U}[\![w^{\pm 1}]\!]\right)[\lambda]$$

Reference: V. G. Kac: *Vertex algebras for beginners* (2nd ed., 1998).

Vertex algebras: Virasoro and affinization

Return to the examples in the Sugawara construction:

$$\mathtt{Vir} = \mathrm{Der}_{\mathbb{C}}(\mathbb{C}[t^{\pm 1}]) \oplus \mathbb{C}\mathtt{C}, \quad \widehat{\mathfrak{g}} = \mathfrak{g}[t^{\pm 1}] \oplus \mathbb{C}\mathtt{K}.$$

- The Virasoro vector $\mathtt{L}(z) = \sum_{n \in \mathbb{Z}} \mathtt{L}_n z^{-n-2} \in \mathtt{Vir}[\![z^{\pm 1}]\!]$ is continuous as a linear map $\mathbb{C}[z^{\pm 1}] \to \mathtt{Vir}$, local with respect to itself, and has OPE
$$[\mathtt{L}_\lambda \mathtt{L}] = (\partial + 2\lambda)\mathtt{L} + \lambda^3 \frac{\mathtt{C}}{12}.$$

Vertex algebras: Virasoro and affinization

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$$\mathtt{Vir} = \mathrm{Der}_{\mathbb{C}}(\mathbb{C}[t^{\pm 1}]) \oplus \mathbb{C}\mathtt{C}, \quad \widehat{\mathfrak{g}} = \mathfrak{g}[t^{\pm 1}] \oplus \mathbb{C}\mathtt{K}.$$

- The Virasoro vector $\mathtt{L}(z) = \sum_{n \in \mathbb{Z}} \mathtt{L}_n z^{-n-2} \in \mathtt{Vir}[\![z^{\pm 1}]\!]$ is continuous as a linear map $\mathbb{C}[z^{\pm 1}] \to \mathtt{Vir}$, local with respect to itself, and has OPE
$$[\mathtt{L}_\lambda \mathtt{L}] = (\partial + 2\lambda)\mathtt{L} + \lambda^3 \frac{\mathtt{C}}{12}.$$
- The 'currents' $\mathtt{a}(z) = \sum_{n \in \mathbb{Z}} (\mathtt{a}t^n) z^{-n-1} \in \widehat{\mathfrak{g}}[\![z^{\pm 1}]\!]$ (for $\mathtt{a} \in \mathfrak{g}$) are continous as linear maps $\mathbb{C}[z^{\pm 1}] \to \widehat{\mathfrak{g}}$, local to each other and have OPEs
$$[\mathtt{a}_\lambda \mathtt{b}] = [\mathtt{a}, \mathtt{b}] + \lambda \langle \mathtt{a}, \mathtt{b} \rangle \mathtt{K} \quad (\mathtt{a}, \mathtt{b} \in \mathfrak{g}).$$

SUSY Lie conformal algebras

A vertex algebra can be seen as a **quantized version of a Poisson algebra**:

- it has a bracket, called 'Λ-bracket', describing the OPEs,
- it has a multiplication, called 'normal ordered product'.

Both operations should satisfy some version of the Leibzniz rule.

Define graded associative algebras (with variables S, χ odd and T, λ even)

$$\mathcal{H} = \frac{\mathbb{C}\langle S, T\rangle}{(S^2 = T)}, \qquad \mathcal{L} = \frac{\mathbb{C}\langle \chi, \lambda\rangle}{(\chi^2 = -\lambda)}.$$

Notation: $\nabla = (T, S)$, $\Lambda = (\lambda, \chi)$ and another copy $\Gamma = (\gamma, \eta)$ of Λ.

SUSY Lie conformal algebras

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Notation: $\nabla = (T, S)$, $\Lambda = (\lambda, \chi)$ and another copy $\Gamma = (\gamma, \eta)$ of Λ.
The most basic piece of a SUSY vertex algebra is the Λ-bracket:

Definition (Heluani–Kac 2007)

A *($N_K = 1$) SUSY Lie conformal algebra* is given by

- an $\mathcal{H}$-module $\mathcal{R}$,
- a parity-reversing Λ-bracket $[\cdot_\Lambda \cdot] \colon \mathcal{R} \otimes \mathcal{R} \longrightarrow \mathcal{L} \otimes \mathcal{R}$,
- axioms ($a, b, c \in \mathcal{R}$):

$$[a_\Lambda b] = (-1)^{|a||b|}[b_{-\Lambda-\nabla} a],$$

$$[a_\Lambda [b_\Gamma c]] = (-1)^{|a|+1}[[a_\Lambda b]_{\Lambda+\Gamma} c] + (-1)^{(|a|+1)(|b|+1)}[b_\Gamma [a_\Lambda c]],$$

$$[Sa_\Lambda b] = \chi[a_\Lambda b], \qquad [a_\Lambda Sb] = -(-1)^{|a|}\,(S + \chi)\,[a_\Lambda b].$$

SUSY vertex algebras

Definition (Heluani–Kac 2007; cf. Barron 2000)

A *($N_K = 1$) SUSY vertex algebra* is given by

- a (SUSY) Lie conformal algebra $(V, [\cdot_\Lambda \cdot])$,
- a normally ordered product $V \times V \longrightarrow V, \quad (a, b) \longmapsto :ab:$,
- a *vacuum vector* $|0\rangle \in V$ that is a right unit for the normally ordered product,
- axioms ($a, b, c \in V$):

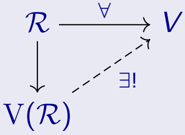
$$:ab: - (-1)^{|a||b|} :ba: = \int_{-\nabla}^{0} d\Lambda [a_\Lambda b],$$

$$:(:ab:)c: - :a(:bc:): = :\left(\int_0^\nabla d\Lambda\, a\right)[b_\Lambda c]: + (-1)^{|a||b|} :\left(\int_0^\nabla d\Lambda\, b\right)[a_\Lambda c]:,$$

$$[a_\Lambda :bc:] = :[a_\Lambda b]c: + (-1)^{(|a|+1)|b|} :b[a_\Lambda c]: + \int_0^\Lambda d\Gamma [[a_\Lambda b]_\Gamma c].$$

Remarks

- Just as a Lie algebra $\mathfrak{g}$ freely generates the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$, a Lie conformal algebra $\mathcal{R}$ freely generates a universal enveloping SUSY vertex algebra $\mathrm{V}(\mathcal{R})$, with a universal property with respect to morphisms into SUSY vertex algebras.



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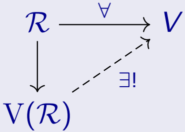

- A SUSY vertex algebra has a canonical *state-field correspondence*

$$Y: V \to \mathrm{End}(V)[\![z^{\pm 1}, \theta]\!], \; a \mapsto a(z,\theta) = \sum_{n\in\mathbb{Z}} z^{-n-1} a_{(n|1)} + z^{-n-1}\theta a_{(n|0)}.$$

The *Fourier modes* $a_{(n|1)}, a_{(n|0)}$ generate a (non-SUSY) vertex algebra, whose λ-bracket is related to the SUSY Λ-bracket via

$$[a_\Lambda b] = \sum_{n\in\mathbb{N}} \frac{\lambda^n}{n!} a_{(n|0)}(b) + \sum_{n\in\mathbb{N}} \frac{\chi\lambda^n}{n!} a_{(n|1)}(b) = [Sa_\lambda b] + \chi[a_\lambda b].$$

Example 1: The ***N*=1 superconformal vertex algebra** is freely generated by an odd superfield $\mathtt{H}$ and an (even) central charge $\mathtt{C}$, i.e.,

$$\mathcal{R} = \mathcal{H} \otimes \mathbb{C}\mathtt{H} \oplus \mathbb{C}\mathtt{C},$$

with non-zero Λ-bracket

$$[\mathtt{H}_\Lambda \mathtt{H}] = (2T + \chi S + 3\lambda)\,\mathtt{H} + \frac{\chi\lambda^2}{3}\mathtt{C}.$$

Expanding in components the corresponding superfield

$$\mathtt{H}(z,\theta) = \mathtt{G}(z) + 2\theta\,\mathtt{L}(z),$$

and the field components in Fourier modes

$$\mathtt{L}(z) = \sum_{n\in\mathbb{Z}} \mathtt{L}_n z^{-2-n}, \quad \mathtt{G}(z) = \sum_{n\in\frac{1}{2}+\mathbb{Z}} \mathtt{G}_n z^{-\frac{3}{2}-n} \in \mathcal{K}(1\,|\,1)[\![z^{\pm 1}]\!],$$

one recovers the usual Virasoro and Neveu–Schwarz commutation relations

$$[\mathtt{L}_m, \mathtt{L}_n] = (m-n)\mathtt{L}_{m+n} + \delta_{m+n,0}(m^3 - m)\frac{\mathtt{C}}{12},$$

$$[\mathtt{L}_m, \mathtt{G}_n] = \left(\frac{m}{2} - n\right)\mathtt{G}_{m+n},$$

$$[\mathtt{G}_m, \mathtt{G}_n] = 2\mathtt{L}_{m+n} + \delta_{m+n,0}\left(m^2 - \frac{1}{4}\right)\frac{\mathtt{C}}{3}.$$

Example 2: The chiral de Rham complex Ω^{ch}_M
(Malikov, Schechtman, Vaintrob 1999)

The **CDR** is a sheaf of vertex algebras over a manifold M that to each open subset $U \subset M$ attaches the vertex algebra $\Omega^{\mathrm{ch}}_M(U)$ generated by

$$f \in C^\infty(U),\ \alpha \in \Omega^1(U),\ \{\mathcal{L}_X, \iota_X \mid X \in \mathrm{Vect}(U)\},$$

subject to the λ-commutators (Song; Linshaw–Mathai 2015)

$$[\mathcal{L}_X{}_\lambda f] = \lambda(Xf), \qquad [\mathcal{L}_X{}_\lambda \iota_Y] = \lambda\iota_{[X,Y]}, \quad [\mathcal{L}_X{}_\lambda \mathcal{L}_Y] = \lambda\mathcal{L}_{[X,Y]},$$
$$[\mathcal{L}_X{}_\lambda \alpha] = \lambda(\mathrm{Lie}_X\,\alpha), \qquad [\iota_X{}_\lambda \alpha] = \lambda(i_X\alpha), \quad [\iota_X{}_\lambda \iota_Y] = 0.$$

and appropriate relations involving normal ordered products (for simplicity, we forget about the SUSY structure).

Example 3: The ***N*=2 superconformal vertex algebra** $\mathrm{V}^{\mathtt{C}}(\mathcal{K}(1|2))$ is freely generated by an odd superfield $\mathtt{H}$, an even superfield $\mathtt{J}$ and an (even) central charge $\mathtt{C}$, i.e.,

$$\mathcal{R} = \mathcal{H} \otimes (\mathbb{C}\mathtt{H} \oplus \mathbb{C}\mathtt{J}) \oplus \mathbb{C}\mathtt{C},$$

with non-zero Λ-brackets $[\mathtt{H}_\Lambda \mathtt{H}] = (2T + \chi S + 3\lambda)\,\mathtt{H} + \frac{\chi\lambda^2}{3}\mathtt{C}$ (as above),

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Expanding in components the corresponding superfields

$$\mathtt{J}(z,\theta) = -i\big(\mathtt{J}(z) + \theta\,(\mathtt{G}^-(z) - \mathtt{G}^+(z))\big),\ \mathtt{H}(z,\theta) = \big(\mathtt{G}^+(z) + \mathtt{G}^-(z)\big) + 2\theta\mathtt{L}(z),$$

and the field components in Fourier modes

$$\mathtt{J}(z) = \sum_{n\in\mathbb{Z}} \mathtt{J}_n z^{-1-n},\ \mathtt{G}^\pm(z) = \sum_{n\in\frac{1}{2}+\mathbb{Z}} \mathtt{G}^\pm_n z^{-\frac{3}{2}-n},$$

one obtains from the above Λ-brackets the commutation relations

$$[\mathtt{J}_m, \mathtt{J}_n] = \frac{m}{3}\delta_{m,-n}c, \quad [\mathtt{J}_m, \mathtt{G}^\pm_n] = \pm\mathtt{G}^\pm_{m+n}, \quad [\mathtt{G}^\pm_m, \mathtt{L}_n] = \left(m - \frac{n}{2}\right)\mathtt{G}^\pm_{m+n},$$

$$[\mathtt{L}_m, \mathtt{J}_n] = -n\mathtt{J}_{m+n},\ [\mathtt{G}^+_m, \mathtt{G}^-_n] = \mathtt{L}_{m+n} + \frac{m-n}{2}\mathtt{J}_{m+n} + \frac{\mathtt{C}}{6}\left(m^2 - \frac{1}{4}\right)\delta_{m,-n}.$$

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Example 4: Let $(\mathfrak{g}, \langle -, - \rangle)$ be a finite-dimensional quadratic Lie algebra, and $\Pi\mathfrak{g}$ the same vector superspace with reversed parity.
The **superaffine vertex algebra** $\mathrm{V}(\mathfrak{g}_{super})$ is freely generated by the SUSY Lie conformal algebra

$$\mathfrak{Cur}\,\mathfrak{g} = \mathcal{H} \otimes \Pi\mathfrak{g} \oplus \mathbb{C}\mathtt{K},$$

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Inspired by physics (Zumino 1979, Álvarez-Gaumé & Freedman 1981, Howe & Papadopoulos 1991...), there is substantial work on embeddings

$$\mathrm{V}^{c}(\mathcal{K}(1|2)) \hookrightarrow H^{0}(M, \Omega_{M}^{\mathrm{ch}}),$$

on the space of sections of

$$\Omega_{M}^{\mathrm{ch}} = \text{chiral de Rham complex on a manifold } M.$$

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- for Calabi–Yau (Malikov, Schechtman & Vaintrob 1999),
- (related) $N = 2$ embeddings for Kähler manifolds (Heluani 2009),
- two $N = 2$ commuting embeddings, i.e., $(2, 2)$ SUSY, for generalized Calabi-Yau manifolds (Heluani & Zabzine 2011),
- other SUSY embeddings on manifolds with special holonomies (e.g., Ekstrand, Heluani, Källen, Zabzine 2013; Rodríguez Díaz 2018...).

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These constructions do not work 'over a point'.
Our approach follows the accepted idea that Ω_M^{ch} **should be viewed as the Courant-version of the superaffine vertex algebra** $\mathrm{V}^{k}(\mathfrak{g}_{super})$.
Therefore we will impose conditions on the Lie algebra $\mathfrak{g}$ inspired by the geometry of Courant algebroids.

Main results

Geometric conditions: the Killing spinor equations

We apply Garcia-Fernandez's theory of generalized connections (inspired by SUGRA) to a (real) finite-dimensional quadratic Lie algebra

$$\mathfrak{g}, \quad \langle -, - \rangle \colon \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathbb{R},$$

viewed as a Courant algebroid over a point.
Fix the following data:

- a generalized metric $V = V_+ \oplus V_-$,
- a 'divergence' $\varepsilon \in V_+$ (viewed as $\langle \varepsilon, \cdot \rangle \in V_+^*$),
- an irreducible representations $S(V_+)$ of Clifford algebra $\mathrm{Cl}(V_+)$.

Choose a basis $\{a_i\}$ of V_+ with dual basis $\{a^i\} \subset V_+$ w.r.t. $\langle -, - \rangle$.

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Definition (Garcia-Fernandez 2019, but restricted to base mnfld=point)

A non-vanishing spinor $\eta \in S(V_+)$ satisfies the *Killing spinor equations* if

$$D_-^+ \eta := \frac{1}{4} \sum_{i,j} \langle [\pi_-, a_i], a^j \rangle a^j a_i \cdot \eta = 0 \qquad \text{(gravitino equation)},$$

$$\not{D}^+ \eta := \frac{1}{12} \sum_{i,j,k} \langle [a_k, a_i], a^j \rangle a^k a^j a_i \cdot \eta - \frac{1}{2} \varepsilon \cdot \eta = 0 \qquad \text{(dilatino equation)}.$$

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Proposition (AC, De Arriba De La Hera, Garcia-Fernandez 2020)

A solution (V_+, ε, η) of the Killing spinor equations on $(\mathfrak{g}, \langle -, - \rangle)$ with $\dim V_+ = 2n$ even and η pure is equivalent to an isotropic decomposition

$$V_+ \otimes \mathbb{C} = \ell \oplus \bar{\ell}$$

(where $\ell = V_+^{1,0}$ and $\bar{\ell} = V_+^{0,1}$), such that

$$[\ell, \ell] \subset \ell, \quad [\bar{\ell}, \bar{\ell}] \subset \bar{\ell} \qquad \text{(F-term equation)},$$

$$\frac{1}{2} \sum_{i=1}^{n} [\epsilon_i, \epsilon^i] = \varepsilon_{\bar{\ell}} - \varepsilon_{\ell} \qquad \text{(D-term equation)},$$

for basis $\{\epsilon_i\} \subset \ell$, $\{\epsilon^i\} \subset \bar{\ell}$ such that $\langle \epsilon_i, \epsilon^j \rangle = \delta_{ij}$.

Main results

Theorem (AC, De Arriba De La Hera, Garcia-Fernandez 2020)

Fix a generalized metric $\mathfrak{g} = V_+ \oplus V_-$ on $(\mathfrak{g}, \langle -, - \rangle)$ with $\dim V_+ = 2n$ and an isotropic decomposition $V_+ \otimes \mathbb{C} = \ell \oplus \bar{\ell}$. Suppose that

$$[\ell, \ell] \subset \ell, \quad [\bar{\ell}, \bar{\ell}] \subset \bar{\ell}, \qquad \sum_{i=1}^{n} [\epsilon_i, \epsilon^i] \in \ell \oplus \bar{\ell},$$

and furthermore,

$$w := \sum_{i=1}^{n} \left([\epsilon_i, \epsilon^i]_\ell - [\epsilon_i, \epsilon^i]_{\bar{\ell}} \right) \in [\ell, \ell]^\perp \cap [\bar{\ell}, \bar{\ell}]^\perp. \qquad (\star)$$

Then there is an embedding $\mathrm{V}^c(\mathcal{K}(1|2)) \hookrightarrow \mathrm{V}^k(\mathfrak{g}^{\mathbb{C}}_{super})$ with central charge $c = 3n$, where $\mathtt{J}, \mathtt{H} \in \mathrm{V}^c(\mathcal{K}(1|2))$ map into

$$\mathtt{J}_0 = \frac{i}{k} {:}e^j e_j{:}, \qquad \mathtt{H}' = \frac{1}{k} \left({:}e_j \left(S e^j\right){:} + {:}e^j \left(S e_j\right){:} \right) + \frac{1}{k} T(\Pi w)$$
$$+ \frac{1}{k^2} \Big({:}e_j \left({:}e^k [e^j, e_k]{:} \right){:} + {:}e^j \left({:}e_k [e_j, e^k]{:} \right){:}$$
$$- {:}e_j \left({:}e_k [e^j, e^k]{:} \right){:} - {:}e^j \left({:}e^k [e_j, e_k]{:} \right){:} \Big),$$

where $e_i := \Pi \epsilon_i \in \Pi \ell$ and $e^i := \Pi \epsilon^i \in \Pi \bar{\ell}$.

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Then there is an embedding $\mathrm{V}^c(\mathcal{K}(1\,|\,2)) \longrightarrow \mathrm{V}^k(\mathfrak{g}^{\mathbb{C}}_{super})$ with central charge $c = 3n$.

Remarks:

1. When $V_+ = \mathfrak{g}$, we recover a classical construction by Getzler 1995 for Manin triples $\mathfrak{g}^{\mathbb{C}} = \ell \oplus \bar{\ell}$, who required precisely the condition $(\star)$.
2. The condition $(\star)$ is satisfied if w is 'holomorphic', i.e., $[w, \ell] \subset \ell$.

Main results

Theorem (AC, De Arriba De La Hera, Garcia-Fernandez 2020)

A solution $(V_+ \otimes \mathbb{C} = \ell \oplus \bar{\ell}, \varepsilon)$ of the Killing spinor equations on $(\mathfrak{g}, \langle -, - \rangle)$ such that $\dim V_+ = 2n$ is even and $\varepsilon \in V_+$ is holomorphic (i.e., $[\varepsilon, \ell] \subset \ell$), induces an embedding

$$\mathrm{V}^c(\mathcal{K}(1|2)) \hookrightarrow \mathrm{V}^k(\mathfrak{g}^{\mathbb{C}}_{super})$$

with central charge $c = 3\big(n + \frac{4}{k}\langle \varepsilon, \varepsilon \rangle\big)$, where $\mathtt{J}, \mathtt{H} \in \mathrm{V}^c(\mathcal{K}(1|2))$ map into

$$\mathtt{J}^{\mathfrak{g}} = \frac{i}{k} :e^j e_j: - \frac{2}{k} S\left(i\Pi(\varepsilon_\ell - \varepsilon_{\bar{\ell}})\right),$$

$$\mathtt{H}^{\mathfrak{g}} = \frac{1}{k}\left(:e_j(Se^j): + :e^j(Se_j):\right) + \frac{1}{k^2}\Big(:e_j\big(:e^k[e^j, e_k]:\big): + :e^j\big(:e_k[e_j, e^k]:\big): - :e_j\big(:e_k[e^j, e^k]:\big): - :e^j\big(:e^k[e_j, e_k]:\big):\Big).$$

Geometric applications

The chiral de Rham complex

We will follow Heluani's superfield approach, based on Bressler 2007.

Theorem (Heluani 2009)

To each Courant algebroid E, one can attach functorially a sheaf of $N = 1$ SUSY vertex algebras $\mathcal{U}^E$ on M, which coincides with the chiral de Rham complex Ω_M^{ch} when E is the standard Courant algebroid $TM \oplus T^*M$.

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Sketch: Let $\mathcal{R}^E$ be the sheaf of $\mathcal{H}$-modules on M freely generated by $C^\infty(M) \oplus \Gamma(\Pi E)$, modulo the relation $\mathcal{D}f = 2Sf$ for $f \in C^\infty(M)$. Then the following formulas induce a structure of sheaf of $N=1$ SUSY Lie conformal algebra on $\mathcal{R}^E$, where $f, g \in C^\infty(M)$ and $a, b \in \Gamma(\Pi E)$:

$$[a_\Lambda b] = [a,b] + 2\chi\langle a,b\rangle \quad \text{(level } k{=}2\text{ !)},$$
$$[a_\Lambda f] = \pi(a)f, \quad [f_\Lambda g] = 0.$$

Then $\mathcal{U}^E$ is constructed as the quotient of the universal enveloping $N=1$ vertex algebra of $\mathcal{R}^E$ modulo the ideal generated by the relations

$$:fg: = fg, \quad :fa: = fa, \quad 1_M = |0\rangle.$$

Twisted Calabi–Yaus

We apply results about **canonical metrics on holomorphic Courant algebroids**.

- Garcia-Fernandez, Rubio, Shahbazi & Tipler, arXiv 2018, Proc. London Math., (to appear)

Fix compact complex manifold X of dim. n with $c_1(X) = 0 \in H^2(X, \mathbb{Z})$.

Recall:

Definition: An $SU(n)$-structure on X is a pair (Ψ, ω) such that

- Ψ is a complex $(n, 0)$-form on X,
- $\omega \in \Omega^{1,1}(X)$ positive $(1,1)$-form, with metric $g = \omega(\cdot, J\cdot)$,
- $\|\Psi\|_g = 1$.

Lee form: $\theta_\omega = Jd^*\omega$.

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Twisted Calabi–Yau eqn. (Garcia-Fernandez, Rubio, Shahbazi & Tipler)**:**

$$\begin{aligned} d\Psi - \theta_\omega \wedge \Psi &= 0, \\ d\theta_\omega &= 0, \\ dd^c\omega &= 0. \end{aligned}$$

Proposition (AC, De Arriba De La Hera, Garcia-Fernandez 2020)

Let K be an even-dimensional compact Lie group. Then a left-invariant solution (Ψ, ω) of the twisted Calabi-Yau equation on K induces a solution of the Killing spinor equations (with $\varepsilon = 2\theta_\omega|_{V_+}$) on the quadratic Lie algebra (with $H = -d^c\omega$)

$$(\mathfrak{g} = \Gamma(TK \oplus T^*K)^K, [\cdot,\cdot]_H, \langle\cdot,\cdot\rangle).$$

Definition (Linshaw–Mathai 2015, after Heluani 2009)

For a closed $H \in \Omega^3(M)$, the *H-twisted chiral de Rham compex* is the ($N = 1$ SUSY) vertex algebra attached to the H-twisted standard Courant algebroid:

$$\Omega_M^{\mathrm{ch},H} := \mathcal{U}^E, \quad \text{with } E = (TM \oplus T^*M, [\cdot,\cdot]_H).$$

Proposition (AC, De Arriba De La Hera, Garcia-Fernandez 2020)

Let K be an even-dimensional compact Lie group that carries a left-invariant solution (Ψ, ω) of the twisted Calabi-Yau equation. Consider the quadratic Lie algebra (with $H = -d^c\omega$)

$$(\mathfrak{g} = \Gamma(TK \oplus T^*K)^K, [\cdot,\cdot]_H, \langle\cdot,\cdot\rangle).$$

If $\theta_\omega^\sharp$ is holomorphic, then the pair (Ψ, ω) induces a vertex algebra embedding

$$\begin{array}{ccccc} \mathrm{V}^c(\mathcal{K}(1\,|\,2)) & \hookrightarrow & \mathrm{V}^2(\mathfrak{g}^{\mathbb{C}}_{super}) & \hookrightarrow & H^0(K, \Omega_K^{\mathrm{ch},H}), \\ \mathtt{J} & \longmapsto & \mathtt{J}^{\mathfrak{g}} & \longmapsto & \mathtt{J}(\Psi,\omega) \\ \mathtt{H} & \longmapsto & \mathtt{H}^{\mathfrak{g}} & \longmapsto & \mathtt{H}(\Psi,\omega) \end{array}$$

in the space of global sections of $\Omega_K^{\mathrm{ch},H}$ with $H = -d^c\omega$.

Example

Recall:

Twisted Calabi–Yau eqn. (Garcia-Fernandez, Rubio, Shahbazi & Tipler) **:**

$$\begin{aligned} d\Psi - \theta_\omega \wedge \Psi &= 0, \\ d\theta_\omega &= 0, \\ dd^c\omega &= 0. \end{aligned}$$

Remarks:

- Pluriclosed metric ($dd^c\omega = 0$) $\Longrightarrow$ positive Aeppli class $[\omega] \in H_A^{1,1}(X, \mathbb{R})$.
  Here, $H_A^{\bullet,\bullet}(X) = \dfrac{\ker \partial\bar{\partial}}{\mathrm{Im}\,\partial + \mathrm{Im}\,\bar{\partial}}$.
- If $[\theta_\omega] = 0 \in H^1(X, \mathbb{R})$, twisted Calabi–Yau eqn. $\Longleftrightarrow$ Kähler Calabi-Yau.

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Theorem (Garcia-Fernandez, Rubio, Shahbazi, Tipler)

Let X be a compact complex surface with $c_1(X) = 0$. Then X admits a solution if and only if one of the following holds:

- $X \cong K3$ or T^4; in this case $[\theta_\omega] = 0 \in H^1(X, \mathbb{R})$, $[d^c\omega] = 0 \in H^3(X, \mathbb{R})$.
- $X \cong \mathbb{C}^2 \backslash \{0\}/\mathbb{Z}$ is a quaternionic Hopf surface; in this case $[\theta_\omega] \neq 0$ and $[d^c\omega] \neq 0$.

If this is the case, then X admits a unique solution on each positive Aeppli class.

Example: $S^3 \times S^1$ viewed as a Lie group $K = SU(2) \times U(1)$, with Lie alg.

$$\mathfrak{k} = \langle v_1, v_2, v_3, v_4 \rangle,$$

where $[v_2, v_3] = -v_1, \quad [v_3, v_1] = -v_2, \quad [v_1, v_2] = -v_3, \quad [v_4, \cdot] = 0.$

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$$v_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \; v_1 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; v_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in \mathfrak{su}(2), \; v_4 = 1 \in \mathfrak{u}(1).$$

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Fix $\ell \in \mathbb{R}_{>0}$. For any $x \in \mathbb{R}_{>0}$, there is a left-invariant solution (Ψ_x, ω_x) of the twisted Calabi–Yau equations

$$d\Psi = \theta_\omega \wedge \Psi, \qquad d\theta_\omega = 0, \qquad dd^c\omega = 0,$$

given in terms of the dual left-invariant differential forms v^1, v^2, v^3, v^4 by

$$\begin{aligned} \omega_x &= \ell x v^{41} + \ell v^{23}, \\ \Psi_x &= \tfrac{\ell}{2}(iv^1 + xv^4) \wedge (v^2 + iv^3), \end{aligned}$$

with

- Lee form $\theta_x = -xv^4$,
- complex structure $I_x \cong x \in \mathbb{R}_{>0}$ given by $I_x v_4 = xv_1$, $I_x v_2 = v_3$,
- $H := -d^c\omega_x = \ell v^{123}$,
- Aeppli class on $X_x = (K, I_x)$: $[\omega_x] \cong a := \ell x \in H^{1,1}_A(X_x, \mathbb{R}) \cong \mathbb{R}$.

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In complex coordinates, $X_x \cong (\mathbb{C}^2 \backslash \{0\})/\{(z_1, z_2) \sim (e^x z_1, e^x z_2)\}$ (Hopf surface).

Proof: By the above bijection between left-invariant solutions to the Tw. CY equations on K and Killing spinors on $\mathfrak{g}$, we get a quadratic Lie algebra

$$\mathfrak{g}_\ell = \mathfrak{k} \oplus \mathfrak{k}^*, \quad \langle v+\alpha, v+\alpha \rangle = \alpha(v),$$
$$[v+\alpha, w+\beta]_\ell = [v,w] - \beta([v,\cdot]) + \alpha([w,\cdot]) + \ell i_w i_v v^{123},$$

with

- generalized metric $V_+^x = \langle v_2 + \ell v^2, v_3 + \ell v^3, v_1 + \ell v^1, v_4 + \ell x^2 v^4 \rangle \subset \mathfrak{g}_\ell$,
- isotropic decomposition $V_+^x \otimes \mathbb{C} = V_+^{x;1,0} \oplus V_+^{x;0,1}$, with

  $$V_+^{x;1,0} = \langle \epsilon_1^+, \epsilon_2^+ \rangle, \quad V_+^{x;0,1} = \langle \overline{\epsilon}_1^+, \overline{\epsilon}_2^+ \rangle,$$

  where

  $$\epsilon_1^+ = \frac{1}{\sqrt{2}}\left(\left(\sqrt{\tfrac{x}{a}} v_2 + \sqrt{\tfrac{a}{x}} v^2\right) - i\left(\sqrt{\tfrac{x}{a}} v_3 + \sqrt{\tfrac{a}{x}} v^3\right)\right), \quad \overline{\epsilon}_1^+ = \overline{\epsilon_1^+},$$
  $$\epsilon_2^+ = \frac{1}{\sqrt{2}}\left(\left(\tfrac{1}{\sqrt{xa}} v_4 + \sqrt{xa}\, v^4\right) - i\left(\sqrt{\tfrac{x}{a}} v_1 + \sqrt{\tfrac{a}{x}} v^1\right)\right), \quad \overline{\epsilon}_1^+ = \overline{\epsilon_1^+}.$$

- I_x-holomorphic divergence $\varepsilon_+^x = -x v_+^4 = -\frac{1}{2}(\frac{1}{a} v_4 + x v^4)$ (in fact, in the centre of $\mathfrak{g}_\ell$),

With these identifications, the proof follows by direct calculation.

Theorem (AC, De Arriba De La Hera, Garcia-Fernandez 2020)

Let $K = SU(2)\times U(1)$ and $\ell\in\mathbb{R}_{>0}$. Consider the family of solutions $(V_+^x, \varepsilon_+^x, l_x)$ of the Killing spinor equations on $\mathfrak{g}_\ell = (\mathfrak{k}\oplus\mathfrak{k}^*, [-,-]_\ell)$, for $x\in\mathbb{R}_{>0}$. Then ε_+^x are holomorphic, so we get vertex-algebra embeddings

$$\mathrm{V}^c(\mathcal{K}(1|2)) \hookrightarrow \mathrm{V}^2(\mathfrak{g}^{\mathbb{C}}_{\ell,super}) \hookrightarrow H^0\left(K, \Omega_K^{\mathrm{ch},H_\ell}\right)$$

$$\mathrm{J} \longmapsto \mathrm{J}^{\mathfrak{g}} \longmapsto \mathrm{J}_x := \mathrm{J}(V_+^x, \varepsilon_+^x, l_x)$$

$$\mathtt{H} \longmapsto \mathtt{H}^{\mathfrak{g}} \longmapsto \mathtt{H}_x := \mathtt{H}(V_+^x, \varepsilon_+^x, l_x)$$

with central charge $c = 6 + 6/\ell$. Furthermore, for all $x\in\mathbb{R}_{>0}$, the solution $(V_+^x, \varepsilon_+^x, l_x)$ is $(0,2)$-mirror to the solution $(V_+^{\hat{x}}, \varepsilon_+^{\hat{x}}, -l_{\hat{x}})$, where

$$\hat{x} = \frac{1}{\ell x}.$$

More precisely, there is a vertex algebra automorphism

$$\psi \in \mathrm{Aut}\left((\Omega_K^{\mathrm{ch},H_\ell})^{\mathfrak{u}(1)}\right),$$

such that

$$\psi(\mathrm{J}_x) = -\mathrm{J}_{\hat{x}}, \quad \psi(\mathtt{H}_x) = \mathtt{H}_{\hat{x}}.$$

N=4 SUSY

Let $K = SU(2)\times U(1)$ and $E_\ell = TK \oplus T^*K$, twisted by $H = \ell v^{123}$ for fixed $\ell \in \mathbb{R}_{>0}$. For each $x \in \mathbb{R}_{>0}$, consider the bi-invariant metric on K

$$g_{\ell,x} = \ell(v^1 \otimes v^1 + v^2 \otimes v^2 + v^3 \otimes v^3 + x^2 v^4 \otimes v^4),$$

corresponding to the bi-invariant generalized metric on E_ℓ given by

$$V_\pm^{\ell,x} = \{v \pm g_{\ell,x}(v) \mid v \in \mathfrak{k}\}.$$

Then $g_x = \omega_{I_x}(\cdot, I_x\cdot) = \omega_{J_x}(\cdot, J_x\cdot) = \omega_{K_x}(\cdot, K_x\cdot)$ is compatible with the left-invariant hyperholomorphic structure (I_x, J_x, K_x) on K given by

$$2\Psi_x = \omega_{J_x} + i\omega_{K_x}.$$

Theorem (AC, De Arriba De La Hera, Garcia-Fernandez 2020)

The triples $(V_+^x, \varepsilon_+^x, I_x)$, $(V_+^x, \varepsilon_+^x, J_x)$, $(V_+^x, \varepsilon_+^x, K_x)$ are left-invariant solutions of the Killing spinor equations on E_ℓ, with fixed Lee form $\theta = -xv^4$. Furthermore, they determine an embedding of the $N = 4$ superconformal vertex algebra of central charge $c = 6$:

$$\mathrm{V}^c(\mathcal{K}(1|4)) \longrightarrow \mathrm{V}^2(\mathfrak{g}_{super}^{\mathbb{C}}) \longrightarrow H^0(K, \Omega_K^{\mathrm{ch},H}),$$

Some open problems:

- Find invariant Killing spinors on other groups and homogeneous spaces M and study corresponding embeddings

$$\mathrm{V}^c\left(\mathcal{K}(1|2)\right) \hookrightarrow \mathrm{V}^2\left(\mathfrak{g}^{\mathbb{C}}_{\ell,super}\right) \hookrightarrow H^0\left(M, \Omega_M^{\mathrm{ch},H_\ell}\right).$$

  Cortés–Krusche's classification of left-invariant generalized connections on 3-dimensional Lie groups may help here!
  Construct examples of (0,2) mirrors.

- More generally find when Killing spinors on Courant algebroids E induce embeddings on arbitrary complex manifolds:

$$\mathrm{V}^c(\mathcal{K}(1|2)) \hookrightarrow H^0\left(M, \mathcal{U}^E\right).$$

Thank you!