# Complex Dirac structures: invariants and local description

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# Outline

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$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi,$$

where $X + \xi, Y + \eta \in \Gamma(\mathbb{T}M)$, called the Courant-Dorfman bracket.

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A **Dirac structure** is a lagrangian subbundle of $\mathbb{T}M$ whose space of sections is closed under the Courant-Dorfman bracket. A **complex Dirac structure** is a Dirac structure on $\mathbb{T}_{\mathbb{C}}M := \mathbb{T}M \otimes \mathbb{C}$.

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## Proposition (I. Vaisman [V])

*Let $N \overset{\iota}{\hookrightarrow} M$ be a submanifold of $M$ and let $L$ be a generalized complex structure on $M$ with associated bundle map*

$$\mathcal{J} = \begin{pmatrix} A & \pi \\ \sigma & -A^* \end{pmatrix}.$$

*Then $\iota^! L$ is a generalized complex structure on $N$ if and only if*

i) *$N$ is a Dirac-Poisson submanifold of $(M, \pi)$.*

ii) *$A(TN) \subseteq TN + \pi(T^*M)|_N = TN \oplus \pi(\mathrm{Ann}\, TN)$.*

iii) *$pr_{TN} \circ A$ is differentiable, where $pr_{TN}$ comes from the projection onto $TN$ of the direct sum of ii).*

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a) If $(M, J)$ is a complex manifold and $\operatorname{codim} S = 1$, then $S$ inherits a CR structure.
b) If $(M, \omega)$ is a symplectic manifold and $\operatorname{codim} S = 1$, then $(S, \iota^*\omega)$ is a presymplectic manifold such that $\dim \ker \iota^*\omega = 1$.

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We have the following.

## Proposition

*Let L be a generalized complex structure and a codimension-one submanifold $S \overset{\iota}{\hookrightarrow} M$. Then*

$$\operatorname{rank} \iota^! L \cap \overline{\iota^! L} = 1.$$

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**Proposition**

*A lagrangian subbundle $L \subseteq \mathbb{T}_{\mathbb{C}}M$ with constant real index $r$, is equivalent to the choice of an $r$-dimensional isotropic subbundle $K \subseteq \mathbb{T}M$ and a bundle map $\mathcal{J} : K^{\perp}/K \to K^{\perp}/K$ such that $\mathcal{J}^2 = -1$ and $\mathcal{J}^* + \mathcal{J} = 0$.*

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$$E = pr_{TM_{\mathbb{C}}} L \subseteq TM_{\mathbb{C}},$$

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**Proposition (M. Gualtieri [G])**

*A lagrangian subbundle $L \subseteq \mathbb{T}M$ has real index zero at a point $p \in M$ if and only if $D|_p = T_pM$ and $\omega_{\Delta|p}$ is nondegenerate.*

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*Let $L$ be a lagrangian subbundle of $\mathbb{T}M$ with real index $r$ at a point $p \in M$. Then*

$$\mathrm{codim}\, D|_p + \dim \ker \omega_{\Delta|p} = r.$$

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a) The order is always zero on generalized complex structures.

b) This definition of type recovers the type of a generalized complex structure when the real index is zero.

c) When the real index is $r$ and $\dim M = 2n + r$,

$$0 \leq \text{type} \leq n.$$

## Lemma

*The real index and the order of a lagrangian subbundle of $\mathbb{T}_{\mathbb{C}}M$ are upper semi-continuous functions. Furthermore, the values of the real index have the same parity.*

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We will see that if $L$ has constant order, then $(\Delta, \omega_\Delta)$ is a presymplectic distribution coming from a (real) Dirac structure.

# Examples

## Example (Regular Dirac structures)

Let $S \subseteq TM$ be an involutive regular distribution and $\omega \in \wedge^2 S^*$ such that $d_S\omega = 0$, i.e. $L(S,\omega)$ is a regular Dirac structure. Then, $L = L(S_{\mathbb{C}}, i\omega_{\mathbb{C}})$ is a complex Dirac structure. The real index of $L$ at $p$ is $\dim \ker \omega|_p + \operatorname{corank} S$ at $p \in M$,

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A **transverse CR structure** is a triple $(R, S, J)$ consisting of two regular distributions $R \subseteq S \subseteq TM$, where $R$ is integrable and a bundle map $J : S/R \to S/R$ such that $J^2 = -Id$ and $q^{-1}(\ker(J_{\mathbb{C}} - iId))$ is involutive on $TM_{\mathbb{C}}$.

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## Example (Transverse CR structure)

Consider the transverse CR structure $(R, S, J)$; $E = q^{-1}(\ker(J_{\mathbb{C}} - iId))$. Since $E$ is involutive in $TM_{\mathbb{C}}$, the lagrangian subbundle $L_{(R,S,J)} = L(E, 0)$ is a complex Dirac structure. The real index of $L_{(R,S,J)}$ is $r = \operatorname{corank} S + \operatorname{rank} R$. Let $n$ be the nonnegative integer such that $\dim M = 2n + r$. We have that

$$\operatorname{order}(L_{(R,S,J)}) = \operatorname{corank} S \qquad \text{and} \qquad \operatorname{type}(L_{(R,S,J)}) = n.$$

# Classification

## Proposition

*Let $L$ be a complex Dirac structure with constant real index $r$ over a $(2n+r)$-dimensional manifold. We have the following:*

a) *If $L$ has order $s$ and type 0, then $L = e^B L(D_{\mathbb{C}}, i\omega_{\mathbb{C}})$, where $L(D_{\mathbb{C}}, i\omega_{\mathbb{C}})$ is the complex Dirac structure associated to $(D, \omega)$ where $D$ is a corank-$r$ involutive distribution and $\omega \in \wedge^2 D^*$ is a presymplectic structure with $(r-s)-$dimensional kernel on each leaf of $D$, and $B$ is a not necessarily closed real two-form such that $d_D B = 0$.*

b) *If $L$ has order $s$ and type $n$, then $L = e^B L(E, 0)$, where $L(E, 0)$ is the complex Dirac structure associated to a transverse CR structure and $B \in \Omega^2(M, \mathbb{C})$ is not necessarily closed.*

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We recover the classification of generalized complex structures on vector spaces.

# Classification up to $B$-transformations

| | | | |
|---|---|---|---|
| order $r$ | $(\Delta, \omega)$ symplectic subspace, codim $\Delta = r$ | | $(D, J)$ codimension-$r$ CR structure |
| order $s$ | $(\Delta, \omega)$ presymplectic subspace, codim $\Delta = s$ $\omega \in \wedge^2 \Delta^*$, $\dim \ker \omega = r - s$ | $(\Delta^{2(n-k)+r-s}, \omega) \times (N^{2k+s}, D, J)$ $(\Delta, \omega)$ presymplectic space $\dim \ker \omega = r - s$ $(D, J)$ codimension-$s$ CR structure on $N$ | $(\Delta_0, D, J)$ transverse CR structure codim $D = s$, $\dim \Delta_0 = r - s$ |
| order 0 | $(V, \omega)$ presymplectic space, $\dim \ker \omega = r$ | | $(\Delta_0, V, J)$ transverse CR structure $\dim \Delta_0 = r$ |
| | type 0 | type $k$ | type $n$ |

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*Let $S \overset{\iota}{\hookrightarrow} M$ be a submanifold. Then, $\iota^! \widehat{L} = \widehat{\iota^! L}$.*

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$$[X + \alpha, Y + \beta]_D = [X, Y] + \mathcal{L}^D_X \beta - \imath_Y d_D \alpha,$$

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Consider the quotient map $q : (D \oplus T^*M)_{\mathbb{C}} \to (D \oplus D^*)_{\mathbb{C}}$. The distribution $q(L)$ is a lagrangian subbundle of $(D \oplus D^*)_{\mathbb{C}}$ with real index zero.

## Lemma

*Let $L$ be a lagrangian subbundle of $\mathbb{T}_{\mathbb{C}}M$ such that $L \cap \overline{L} = (\operatorname{Ann} D)_{\mathbb{C}}$. If $D$ is involutive, then $L$ is involutive if and only if $q(L)$ is involutive.*

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*Let $L$ be a complex Dirac structure with constant real index equal to its order such that $L \cap \overline{L} = (\operatorname{Ann} D)_{\mathbb{C}}$. If the distribution $D$ is involutive and $S$ is a leaf of $D$, then $q(L)|_S$ is a generalized complex structure over $S$.*

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This example appeared in the work of D. Li-Bland [L] with the name of a generalized CR structure.

# Splitting theorem

## Theorem (Splitting theorem for Dirac structures, C. Blohmann [Bl])

*Let $L$ be a Dirac structure on $M$ and $p \in M$. Let $N \overset{\iota}{\hookrightarrow} M$ be a submanifold containing $p$, such that $T_pN$ is complement to $P = pr_{TM}L|_p$. Then, there exist a neighbourhood $U$ of $p$, a two-form $B \in \Omega^2_{cl}(P \times N)$ such that*

$$L|_U \cong e^B(L' \times L_\omega),$$

*where $L_\omega$ is the Dirac structure associated to the presymplectic leaf passing through $p$ and $L' = \iota^! L = Graph(\pi)$ for some Poisson structure $\pi$ over $N$ vanishing at $p$.*

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**Theorem (Splitting theorem for generalized complex structures, M. Abouzaid-M. Boyarchenko [AB])**

*Let $L$ be a generalized complex structure and let $p \in M$. Then, there exists a neighborhood $U$, a closed two-form $B$, a symplectic structure $\omega$ and a generalized complex structure $L'$ such that*

$$L|_U \cong e^B(L' \times L_{i\omega}).$$

*Moreover, the Poisson structure associated to $L'$ vanishes at $p$.*

**Theorem**

*Let $L$ be a complex Dirac structure of $\mathbb{T}_{\mathbb{C}}M$ with constant real index $r$ and order $s$ and a point $p \in M$ of type $k$. Consider a $(2k+s)$-dimensional submanifold $N \subseteq U \overset{\iota}{\hookrightarrow} M$ of complementary dimension to the presymplectic leaf passing through and transversal to $\hat{L}$ at $p$, i.e. $T_pN \oplus \Delta|_p = T_pM$. Then there exist a neighbourhood $U$ of $p$ and a closed real two-form $B$ defined on $U$, such that*

$$L|_U \cong e^B(\iota^! L \times L_{i\omega}),$$

*where $\iota^! L$ is a complex Dirac structure with constant real index $s$ and order $s$ and having associated Poisson bivector vanishing at $p$, $L_{i\omega}$ is the complex Dirac structure associated to the presymplectic leaf $S$ passing through $p$.*

## Corollary

*Let $L$ be a complex Dirac structure with constant real index $r$ and order $s$ and let $p$ be a regular point of type $k$. Then there exist a neighbourhood $U$ of $p$ such that*

$$L|_U \cong e^B(L_{(D,J)} \times L_{i\omega_{can}}),$$

*where $L_{i\omega_{can}}$ is the graph of the canonical presymplectic structure on $\mathbb{R}^{2(n-k)+r-s}$ with kernel of dimension $r-s$, $L_{(D,J)}$ is the complex Dirac structure associated to a CR structure of codimension $s$ over $N$ and $B$ is a real two-form on $M$ which is closed on the directions of $\mathbb{R}^{2(n-k)+r-s}$.*

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Thanks!