Twistor space and parabolic structures

Carlos Simpson

CNRS, Université Côte d'Azur

Moduli spaces and geometric structures
ICMAT, September 12-16, 2022

Dedicated in honor of Oscar García-Prada's 60th birthday

Parabolic structures—some of Oscar's work

O. García-Prada, P. Gothen, V. Muñoz: Betti numbers of the moduli space of rank 3 parabolic Higgs bundles

This was the first major calculation of Betti numbers of moduli spaces after Hitchin's paper.

O. García-Prada, M. Logares, V. Muñoz: Moduli spaces of parabolic $U(p,q)$-Higgs bundles

This relates parabolic structures with certain flat bundles preserving indefinite Hermitian forms, and continues the discussion of topology of moduli spaces. I think it will be very interesting to examine further this theory, to look at flat $U(p,q)$ bundles with monodromy transformations in special position with respect to the indefinite form (for example, eigenspaces that are isotropic).

Parabolic structures—some of Oscar's work

O. Biquard, O. García-Prada, I. Mundet i Riera: Parabolic Higgs bundles and representations of the fundamental group of a punctured surface into a real group

They adapt the theory of parabolic (even parahoric) bundles to the case of principal G-bundles, and obtain the nonabelian Hodge correspondence. The calculations we are going to do below, are done here in the very general setting of G-bundles.

Parabolic structures—some of Oscar's work

O. García-Prada, G. Wilkin: Action of the mapping class group on character varieties and Higgs bundles

O. Biquard, O. García-Prada: Parabolic vortex equations and instantons of infinite energy

These are also very interesting papers that need to be exploited more. Few authors have considered the parabolic vortex equations so far, so this is a quite unexplored and rich territory.

Introduction to parabolic structures

Let us work on a compact curve X with a divisor $D = \{x\}$ consisting of a single point for simplicity.
One way to define a parabolic structure is as follows. Suppose we are given a vector buundle E over X. A *quasiparabolic structure* for E is a filtration

$$0 = F_0 \subset F_1 \subset \ldots \subset F_{k-1} \subset F_k = E_x.$$

A *parabolic structure* consists of a quasiparabolic structure together with a choice of *parabolic weights*

$$-1 < a_1 < \ldots < a_k \leq 0.$$

We may think of a_i as corresponding to the associated graded piece F_i/F_{i-1} of the filtration.

Introduction to parabolic structures

These are related to metrics on E in the following way. We suppose given an hermitian metric h with the property that for any section e of E such that $e(x)$ doesn't vanish in E_x, then the norm should satisfy

$$|z| < |e|_h \leq |z|^{-\varepsilon}$$

(with appropriate constants for any $\varepsilon > 0$). The metric is said to be adapted to a parabolic structure if we have, whenever $e(x) \in F_i - F_{i-1}$,

$$|z|^{-a_i+\varepsilon} < |e|_h < |z|^{-a_i-\varepsilon},$$

in other words $F_i - F_{i-1}$ are the values of sections whose growth rate is approximately $|z|^{-a_i}$.

Introduction to parabolic structures

For sufficiently nice metrics, such as harmonic metrics on tame Higgs bundles, there exists a unique parabolic structure that reflects the growth rate of the metric in this way.

Indeed, we can look at the subsheaf E_a of sections e whose growth is $|e|_h < |z|^{-a-\varepsilon}$ and for $-1 < a \leq 0$ these will restrict to the desired filtration on E_x.

Introduction to parabolic structures

A tame parabolic Higgs bundle is a pair (E, φ) where

$$\varphi : E \to E \otimes K_X(D) = E \otimes \Omega^1_X(\log D)$$

together with a parabolic structure such that the residue of φ, which is a well-defined endomorphism of E_x, respects the filtration. We say that it is *strictly parabolic* if the residue descends the filtration by at least one step.

Introduction to parabolic structures

We define the *associated-graded space*

$$\mathrm{gr}_x(E) := \bigoplus_{i=1}^{k} F_i/F_{i-1}$$

and the *graded residue* denoted $\mathrm{res}_x(\varphi)$ is defined as the action of the residue of φ on the associated-graded space.
The strictly parabolic condition is equivalent to asking that $\mathrm{res}_x(\varphi) = 0$.

We note that this action might not be equivalent to the full residue, indeed the Jordan blocks of the full residue might be longer than those of $\mathrm{res}_x(\varphi)$. However, it is the graded residue that reflects the monodromy of the associated flat connection (or more precisely, a similar graded residue there).

Some calculations in the rank 1 case

Let's do some calculations in the unitary gauge. We fix a trivial $\mathcal{C}^\infty$ line bundle L with constant unitary metric, and consider a unitary connection

$$d_A = d + \frac{a}{2}\left(\frac{d\overline{z}}{\overline{z}} - \frac{dz}{z}\right), \quad a \in \mathbb{R}.$$

We are doing a gauge-fixing so that the radial part of the connection is trivial, and for our toy example we assume rotational symmetry.

Some calculations in the rank 1 case

Consider a Higgs field

$$\Phi = P\frac{dz}{z}, \quad P \in \mathbb{C}.$$

Since Φ is holomorphic and d_A is flat, these give solutions of Hitchin's equations, noting that the $\Phi\Phi^\dagger + \Phi^\dagger\Phi$ term vanishes since it is a commutator and we are in rank 1.

The resulting flat connection is

$$D = d_A + \Phi + \Phi^\dagger = d + \frac{a}{2}\left(\frac{d\overline{z}}{\overline{z}} - \frac{dz}{z}\right) + P\frac{dz}{z} + \overline{P}\frac{d\overline{z}}{\overline{z}}.$$

Some calculations in the rank 1 case

It will be useful to have a general formula to get a holomorphic frame for an operator of the form

$$\overline{\partial}_q = \overline{\partial} + q\frac{d\overline{z}}{\overline{z}}, \quad q \in \mathbb{C}.$$

One may calculate that the function $f = (z\overline{z})^{-q}$ is single valued and satisfies $\overline{\partial}_q(f) = 0$.
The growth rate of this section is $|f| = |z|^{-2\mathrm{Re}q}$.

Some calculations in the rank 1 case

If we use $\overline{\partial}_q = (d_A)^{0,1}$ we have $q = a/2$ and our frame is

$$f = (z\overline{z})^{-a/2}$$

with growth rate $|z|^{-a}$. This means that the parabolic weight of the Higgs bundle is a.

Some calculations in the rank 1 case

The monodromy of the connection is the exponential of

$$\oint (P - a/2)\frac{dz}{z} + (\overline{P} + a/2)\frac{d\overline{z}}{\overline{z}} = 2\pi i(P - a - \overline{P}) = 2\pi \mathrm{Im}(P) - 2\pi i a.$$

Thus, the monodromy has a rotational component determined by a, and a scalar real component determined by the imaginary part $\mathrm{Im}(P)$.

Some calculations in the rank 1 case

One may calculate that the real part $\mathrm{Re}(P)$ determines the growth rate of flat sections along a ray going to the origin.
This may be formalized in the notion of a filtered local system, which is to say a local system provided with a sort of parabolic structure determining the polynomial growth rates of norms of flat sections.

Some calculations in the rank 1 case

We may also look at the holomorphic flat connection, or more generally the λ-connection for $\lambda \in \mathbb{A}^1$. Put

$$D' = \partial_A + \Phi^\dagger, \quad D'' = \overline{\partial}_A + \Phi, \quad D^\lambda = \lambda D' + D''.$$

Thus,

$$D^\lambda = \lambda\partial + \overline{\partial} + \left((P - \lambda a/2)\frac{dz}{z} + (\lambda\overline{P} + a/2)\frac{d\overline{z}}{\overline{z}} \right).$$

Some calculations in the rank 1 case

As we have seen before, this bundle with holomorphic structure $(D^{\lambda})^{0,1}$ has a holomorphic frame of the form $f = (z\bar{z})^{-q}$, $q = (a/2 + \lambda\overline{P})$, having growth rate $|z|^{-b_\lambda}$ with parabolic weight

$$b_\lambda = a + 2\mathrm{Re}(\lambda\overline{P}).$$

Some calculations in the rank 1 case

We note that

$$D^{\lambda}(f)^{1,0} = (-\lambda q + P - \lambda a/2) f \frac{dz}{z}$$

so the resulting holomorphic bundle has a logarithmic λ-connection with residue

$$B_{\lambda} = P - \lambda a - \lambda^2 \overline{P}.$$

At $\lambda = 0$ we recover the Higgs bundle case $b_0 = a$ and $B_0 = P$.

At $\lambda = 1$ the parabolic weight is $b_1 = a + 2\mathrm{Re}(P)$ and the residue is $B_1 = 2i\mathrm{Im}(P) - a$.

Summary

One can show that these calculations on the toy examples do indeed provide local models for the rank 1 pieces of tame solutions of Hitchin's equations. Modeling the strings in the Jordan normal form is a little more complicated, and introduces a modification of size $|\log r|^n$ in the metric, so that doesn't change the polynomial-in-r growth rates.

Summary

A solution of Hitchin's equations yields, for any $\lambda \in \mathbb{A}^1$, a holomorphic logarithmic λ-connection that locally has the form

$$\nabla_h^\lambda = \lambda d + B^\lambda \frac{dz}{z} + \dots$$

and the metric $h \sim |z|^{-b_\lambda}$ where

$$b_\lambda = a + 2\mathrm{Re}(\lambda \overline{P}), \quad B_\lambda = P - \lambda a - \lambda^2 \overline{P}.$$

These equations, due originally to Sabbah and Mochizuki, are well defined up to an operation of $B_\lambda \mapsto B_\lambda - \lambda k$ and $b_\lambda \mapsto b_\lambda + k$ for $k \in \mathbb{Z}$, corresponding to the different choices of extension of the holomorphic line bundle across the origin.

Observations

From this toy situation we can make a few general observations:

- In a rank 1 piece, the local structure is governed by 3 real parameters.
- These parameters group naturally into one complex and one real parameter.
- However, that splitting is different for the Higgs field and for the monodromy or λ-connections.

What is going on and how to explain this?

An explanation?

We would like to show that this somewhat strange looking collection of transformations is really a manifestation of an analogue of Hitchin's quaternionic structure, but in a way that corresponds to Hodge weight 2 rather than Hodge weight 1. Let's go back and look at the twistor space of a quaternionic manifold.

The twistor $\mathbb{P}^1$

Let $\mathbb{H} = \mathbb{R}\langle 1, I, J, K\rangle$ be the algebra of quaternions. A complex structure $\kappa \in \mathbb{H}$, $\kappa^2 = -1$ has the form

$$\kappa = xI + yJ + zK, \quad x^2 + y^2 + z^2 = 1,$$

so the space of complex structures in $\mathbb{H}$ is identified with the two-sphere S^2. This itself is provided with a complex structure making it into $\mathbb{P}^1$.

The complex structure I corresponds to $0 \in \mathbb{P}^1$ and J corresponds to $1 \in \mathbb{P}^1$.

The twistor space of a quaternionic vector space

Suppose V is an $\mathbb{H}$-module i.e. quaternionic vector space. Then for each $\kappa \in \mathbb{P}^1$ we obtain a complex structure on V, hence a $\mathbb{C}$-vector space V_κ.

On the product $V \times \mathbb{P}^1$, declaring the complex structure in the second variable to be the same as that of $\mathbb{P}^1$, this yields a global complex structure making it into the total space of a vector bundle $\mathcal{V}$ over $\mathbb{P}^1$.

Preferred sections and involution

We call the sections of $\mathcal{V}$ of the form $\{v\} \times \mathbb{P}^1$ the *preferred sections* .

The antipodal involution $(x, y, z) \mapsto (-x, -y, -z)$ corresponds to $\sigma : \lambda \mapsto -\overline{\lambda}^{-1}$ (composition of the three natural involutions in coordinates on $\mathbb{P}^1$).

This involution may be viewed as a real structure on $\mathbb{P}^1$ although its set of real points is empty.

Preferred sections and involution

The vector bundle $\mathcal{V}$ is provided with an involution defined by $\sigma(v, \lambda) = (v, \sigma(\lambda))$, antilinear and covering σ on the base. In other words, the bundle has a real structure over the real structure of the base.

The preferred sections are the only holomorphic sections compatible with σ.

In the case $V = \mathbb{H}$ one calculates $\mathcal{V} \cong \mathcal{O}_{\mathbb{P}1}(1)^{\oplus 2}$. Since any $\mathbb{H}$-module is just a direct sum of these, it follows in general that the bundle $\mathcal{V}$ is going to be semistable of slope 1.

Weight 1 property and quaternionic structure

Proposition

If $(\mathcal{V}, \sigma)$ is a vector bundle with σ-compatible involution on $\mathbb{P}^1$ such that $\mathcal{V}$ is semistable of slope 1, then it comes from the above construction for the quaternionic vector space

$$V = \Gamma(\mathbb{P}^1, \mathcal{V})^\sigma.$$

For each $\kappa \in \mathbb{P}^1$, the projection $V \to \mathcal{V}_\kappa$ is an isomorphism of real vector spaces and this induces the complex structure κ on V.

Weight 1 property and quaternionic structure

The basic idea is that when a bundle $\mathcal{V}$ is semistable of slope 1, then for any two points $x, y \in \mathbb{P}^1$ the evaluation map is an isomorphism

$$\Gamma(\mathbb{P}^1, \mathcal{V}) \xrightarrow{\cong} \mathcal{V}_x \oplus \mathcal{V}_y.$$

In other words, we can specify the values of a section at any two distinct points.

Weight 1 property and quaternionic structure

Choosing antipodal points $\kappa, \sigma\kappa$, we find that there is a unique σ-invariant section with a given value at a single point κ. Thus

$$\Gamma(\mathbb{P}^1, \mathcal{V})^\sigma \xrightarrow{\cong} \mathcal{V}_\kappa.$$

The point κ therefore determines a complex structure on the vector space $V = \Gamma(\mathbb{P}^1, \mathcal{V})^\sigma$, and this family of complex structures is the sphere of complex structures of the quaternionic structure.

Twistor space of a quaternionic manifold

Suppose now that we are given an integrable quaternionic manifold M. Then the *twistor space* is

$$Tw(M) := M \times \mathbb{P}^1$$

with complex structure obtained in the same way.

The horizontal "preferred sections" $\{m\} \times \mathbb{P}^1$ are given the holomorphic structure of $\mathbb{P}^1$, and for any $\kappa \in \mathbb{P}^1$ the fiber $M \times \{\kappa\}$ is given the complex structure M_κ determined by the action of $\kappa \in \mathbb{H}$ on the tangent spaces of M.

The "integrability" condition says that these are integrable complex structures, and the general Penrose theory yields an integrable complex structure on the total space $Tw(M)$.

Hitchin's hyperkähler structure

Hitchin defined a quaternionic structure on the moduli space M of local systems on a smooth compact Riemann surface X, and considered the twistor space $Tw(M)$.

The quaternionic structure is in fact a hyperkähler structure having a Riemannian metric that is Kähler for each complex structure.

The twistor space forms a central structure of nonabelian Hodge theory.

Hitchin's hyperkähler structure

For the complex structure $\kappa = I$ corresponding to $\lambda = 0$ in $\mathbb{P}^1$, the complex moduli space M_0 is the moduli space of Higgs bundles (E, φ) on X.

For the complex structure $\kappa = J$ corresponding to $\lambda = 1$ in $\mathbb{P}^1$, the complex moduli space M_1 is the moduli space of vector bundles with integrable connection (E, ∇) on X.

Deligne's interpretation

Furthermore, for all the complex structures κ corresponding to $\lambda \neq 0, \infty$ the moduli spaces M_κ are naturally isomorphic to the moduli space of local systems or representations of the fundamental group.

Deligne gave an interpretation of the twistor space in which M_λ is viewed as the moduli space of *vector bundles with λ-connection* (E, ∇).

Deligne's interpretation

For $\lambda \neq 0$ the rescaling $\lambda^{-1}\nabla$ is just a connection, whereas for $\lambda = 0$ a λ-connection is the same thing as a Higgs field φ.

We may make an algebraic geometry construction of the family of moduli spaces over $\mathbb{A}^1 \subset \mathbb{P}^1$.

In a motivic language, this moduli space corresponds to the *nonabelian Hodge filtration* on the de Rham cohomology space M_1, so we denote the total space by M_{Hod}.

Deligne glueing

We have an analytic trivialization

$$M_{\mathrm{Hod}}|_{\mathbb{G}_m} \cong \mathbb{G}_m \times M_B$$

where M_B is the "Betti" moduli space of representations of the fundamental group.

Now, the topological identification

$$\pi_1(X) \cong \pi_1(\overline{X}) \quad \text{whence} \quad M_B(X) \cong M_B(\overline{X})$$

allows Deligne to define a *glueing* between $M_{\mathrm{Hod}}(X)$ and $M_{\mathrm{Hod}}(\overline{X})$, using the map $\lambda \mapsto -\lambda^{-1}$ on the $\mathbb{G}_m$ coordinate.

Deligne glueing

Glueing the two pieces together yields the *Deligne-Hitchin* space

$$M_{\mathrm{Hod}}(X) \cup M_{\mathrm{Hod}}(\overline{X}) \quad =: \quad M_{\mathrm{DH}} \to \mathbb{P}^1.$$

This definition is made so that we can define an antipodal involution covering σ, using that $M_{\mathrm{Hod}}(\overline{X})$ is the complex conjugate variety to $M_{\mathrm{Hod}}(X)$.

Isomorphism

Theorem

The Deligne-Hitchin moduli space constructed by Deligne's glueing is isomorphic to the twistor space for Hitchin's quaternionic structure:

$$\begin{array}{ccccc} M_{\mathrm{DH}} & & \cong & & Tw(M) \\ & \searrow & & \swarrow & \\ & & \mathbb{P}^1 & & \end{array}$$

Preferred sections from harmonic bundles

The preferred sections of the twistor space correspond to sections of the fibration $M_{\mathrm{DH}} \to \mathbb{P}^1$ that we also call "preferred sections". These are maps $\rho : \mathbb{P}^1 \to M_{\mathrm{DH}}$ that are obtained whenever we have a harmonic bundle

$$(E, \partial, \overline{\partial}, \varphi, \varphi^\dagger, h)$$

corresponding to a solution of Hitchin's equations.

Preferred sections from harmonic bundles

For $\lambda \in \mathbb{A}^1$ the point $\rho(\lambda) \in M_\lambda$ in the moduli space of holomorphic vector bundles with λ-connections is

$$\left(\mathcal{E}_\lambda := (E, \overline{\partial} + \lambda\varphi^\dagger), \;\; \nabla_\lambda := \lambda\partial + \varphi\right).$$

Preferred sections are compatible with the antipodal involution.

Weight 1 property

The fact that this construction gives the twistor space of a quaternionic manifold comes from the following property:

Proposition

Suppose $\rho : \mathbb{P}^1 \to M_{\mathrm{DH}}$ *is a preferred section coming from a harmonic bundle. Let*

$$\mathcal{V} = \rho^* T(M_{\mathrm{DH}}/\mathbb{P}^1)$$

be the pullback of the relative tangent bundle, or equivalently the normal bundle of M_{DH} *to the section. Then* $\mathcal{V}$ *is a semistable vector bundle of slope* 1 *over* $\mathbb{P}^1$.

Weight 1 property

An analogy with Hodge structures motivates us to call the property of being semistable of slope 1, a property of *weight* 1.

This weight 1 property is the nonabelian cohomology analogue of the statement in usual Hodge theory that $H^1_{\mathrm{dR}}(X)$ has a weight 1 Hodge structure, and similarly for a variety over $\mathbb{F}_q$ that the étale cohomology $H^1_{\mathrm{et}}(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell)$ has weight 1 in the sense that the eigenvalues of Frobenius have norm $q^{1/2}$.

Weight 2: analogy with MHS and $\mathbb{Q}_\ell$ cohomology

Recall that the first cohomology of an open variety can have a weight 2 part. For example when

$$X = \mathbb{G}_m = \mathbb{P}^1 - \{0, \infty\}$$

then the mixed Hodge structure on $H^1(X)$ is one-dimensional, of weight 2, and in the arithmetic setting

$$H^1_{\text{et}}(X, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(1)$$

is a Tate twist having weight two.

For more general quasiprojective curves, the weight two part corresponds to the punctures.

Weight 2

We may therefore naturally conjecture that the twistor weight 1 property would become a weight 2 property for the local monodromy transformations at punctures.

Our first remarks are that this holds for local systems of rank 1 over a quasiprojective curve.

The case of rank 2 will be discussed later, and I don't yet know how to do the theory for higher rank.

The weight 2 property in the case of rank 1

For the rank 1 case, let's look at $X = \mathbb{P}^1 - \{0, \infty\}$ where the only data of a local system is a single local monodromy at a puncture.

A logarithmic λ-connection will be over a trivial bundle on X, having the form

$$\nabla(\lambda, a) = \lambda d + a\frac{dz}{z}.$$

For the moduli space we may write

$$M_{\mathrm{Hod}} := \mathbb{A}^1 \times \mathbb{C}/\mathscr{G}$$

using $\mathbb{A}^1$ for the λ variable and $\mathbb{C}$ for the coefficient a.

The weight 2 property in the case of rank 1

Change of the trivialization makes (λ, a) equivalent to $(\lambda, a + k\lambda)$ for any $k \in \mathbb{Z}$. Let $\mathscr{G} \cong \mathbb{Z}$ be this "gauge group" acting.

The group acts discretely over $\lambda \neq 0$ but the stabilizer group of the fiber over $\lambda = 0$ is the full $\mathscr{G} = \mathbb{Z}$. It is therefore not completely clear what kind of structure best to accord to the quotient.

The weight 2 property in the case of rank 1

The Riemann-Hilbert corresponce over $\lambda \neq 0$ is the exponential

$$\mathbb{G}_m \times \mathbb{C}/\mathscr{G} \xrightarrow{\cong} \mathbb{G}_m \times \mathbb{C}^*$$

sending (λ, a) to $(\lambda, \exp(2\pi i a/\lambda))$. It is of course also the map taking a connection to its monodromy representation.

We would like to glue M_{Hod} to the other piece in the Deligne glueing using Riemann-Hilbert.

The weight 2 property in the case of rank 1

Let $\mu = -\lambda^{-1}$ denote the coordinate of the other chart $\mathbb{A}^1 \subset \mathbb{P}^1$. A point

$$(\mu, b) \in M_{\mathrm{Hod}}(\overline{X})$$

has monodromy transformation $\exp(2\pi i b/\mu)$ along the generating loop for $\pi_1(\overline{X})$.

As $X^{\mathrm{top}} \cong \overline{X}^{\mathrm{top}}$ takes the generator to minus the generator, the Deligne glueing should associate (λ, a) with (μ, b) (up to the $\mathscr{G}$ action) when

$$\exp(2\pi i a/\lambda) = \exp(-2\pi i b/\mu).$$

The weight 2 property in the case of rank 1

Lift over the action of the gauge group to say that we would like

$$a/\lambda = -b/\mu.$$

Recalling that $\mu = -\lambda^{-1}$, this associates (λ, a) with $(-\lambda^{-1}, b)$ when

$$a/\lambda = -b/(-\lambda^{-1}) = \lambda b \quad \Leftrightarrow \quad a = \lambda^2 b.$$

This is the glueing condition for the line bundle $\mathcal{O}_{\mathbb{P}^1}(2)$ over $\mathbb{P}^1$. Hence,

$$M_{\mathrm{DH}} = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(2))/\mathscr{G}.$$

The weight 2 property in the case of rank 1

There is also a natural antipodal involution, and the *preferred sections* are the sections that are compatible with σ. These are going to lift over the action of the gauge group to σ-equivariant sections of $\mathcal{O}_{\mathbb{P}^1}(2)$, so in what follows we'll sometimes ignore the $\mathscr{G}$-action.

Recall from the compact case that we wanted to look at the space of σ-equivariant sections of $M_{\mathrm{DH}}/\mathbb{P}^1$. Here let's lift and look at the space of σ-equivariant sections of $\mathcal{O}_{\mathbb{P}^1}(2)$. Recall that before asking for σ-equivariance we have

$$\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) \cong \mathbb{C}^3.$$

The weight 2 property in the case of rank 1

Lemma

$$\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))^\sigma \cong \mathbb{R}^3.$$

For any $\kappa \in \mathbb{P}^1$ the restriction morphism from these sections to the fiber $\mathbb{C}_\kappa = \mathcal{O}_{\mathbb{P}^1}(2)_\kappa$ is a surjection

$$\mathbb{R}^3 \to \mathbb{C}_\kappa.$$

There is a natural splitting as $\mathbb{R}^3 \cong \mathbb{R} \times \mathbb{C}_\kappa$ such that the generator of the gauge group has the form $(1, \lambda)$.

One may identify a general weight 2 structure with a specific one, namely the tangent bundle of $\mathbb{P}^1$ itself, as a way to determine the splitting.

Parabolic weight parameter

The extra real parameter, kernel of the restriction map, may be viewed as the *parabolic weight parameter*.

If $(E, \partial, \overline{\partial}, \varphi, \varphi^\dagger, h)$ is a tame harmonic bundle over X then it yields a σ-invariant section, and for $\lambda \in \mathbb{A}^1$ the corresponding point in $\mathbb{R} \times \mathbb{C}$ is (p, e) where p is the parabolic weight and e the eigenvalue of the residue of the λ-connection.

The parabolic weight expresses the growth rate of the harmonic metric h near a puncture.

Parabolic weight parameter

The expression of $\mathbb{R} \times \mathbb{C}$, depending on λ, as corresponding to a unique $\mathbb{R}^3$ independent of λ, allows to recover the formulas (see below) for the relationship between parabolic weights and eigenvalues of the residue between Higgs bundles and flat bundles.

The fact that we have an extra real parameter for the parabolic weight, may therefore be seen as a manifestation of the fact that the monodromy transformations around punctures lie in a space whose Hodge weight is 2.

Formulas

Sabbah and Mochizuki gave formulas for the variation of parabolic weight $\mathfrak{p}$ and eigenvalue $\mathfrak{e}$ as a function of λ, generalizing the case $\lambda = 1$. We have seen these in the calculations at the beginning.

Starting with $(a, P) \in \mathbb{R} \times \mathbb{C}$ at $\lambda = 0$, we have:

$$\mathfrak{p}(\lambda, (a, P)) = a + 2\mathrm{Re}(\lambda \overline{P})$$

$$\mathfrak{e}(\lambda, (a, P)) = P - a\lambda - \overline{P}\lambda^2.$$

These enter into Mochizuki's discussion of KMS-structure and what he calls "difficulty (b)".

The weight 2 twistor space interpretation leads to the same formulas.

Formulas

The antipodal involution is $\lambda \mapsto -\overline{\lambda}^{-1}$. The action on sections of $\mathcal{O}_{\mathbb{P}1}(2)$, viewed as polynomials of order 2, may be written as

$$f^{\sigma}(\lambda) = -\lambda^2 \overline{f(-\overline{\lambda}^{-1})}.$$

If $f(\lambda) = a + b\lambda + c\lambda^2$ we get

$$f^{\sigma}(\lambda) = -\lambda^2 \overline{\left(a - b\overline{\lambda}^{-1} + c\overline{\lambda}^{-2}\right)}$$

$$= -\lambda^2 \left(\overline{a} - \overline{b}\lambda^{-1} + \overline{c}\lambda^{-2}\right)$$

$$= -\overline{a}\lambda^2 + \overline{b}\lambda - \overline{c}.$$

Formulas

The σ-invariant sections are those where $\overline{a} = -c$ and $b = \overline{b}$. Changing a to P and b to a, we can write such a section as

$$f(\lambda) = P - a\lambda - \overline{P}\lambda^2.$$

We obtain the formula given above for the change of residues as a function of λ.

For the parabolic weight parameters, the space of vectors with normalized length in the real kernel line corresponding to points of $\mathbb{P}^1$, provides a sphere and hence a natural inner product. The parabolic weight splitting is made using this sphere; the calculation is therefore longer but we also obtain the previous formulas for the variation of parabolic weight.

Going towards the higher-rank case

We would like to extend this picture to higher rank local systems on an open curve. Let's look at some potential difficulties in light of the previous discussion.

A first observation is that the action of the gauge group becomes singular over $\lambda = 0$, indeed the entire $\mathscr{G} = \mathbb{Z}$ stabilizes the full fiber over $\lambda = 0$.

For this reason, we'll tend not to really look at the quotient space, which would be some kind of stack or perhaps "diffeological space", but to retain just the action groupoid instead.

Going towards the higher-rank case

A next observation is that we have sidestepped any discussion of stability. From the compact case recall that the construction of M_{Dol} requires the notion of stability of a Higgs bundle, so this information is needed in the fiber over $\lambda = 0$ for the construction of M_{Hod}.

However, in the quasiprojective case, defining stability requires knowing the parabolic weights, but we are trying to recover the parabolic weights from the twistor space construction itself as happened in rank 1.

Without a notion of stability, we are going to be getting moduli spaces that are not of finite type but only locally of finite type. This should be accepted.

Going towards the higher-rank case

A third difficulty for making the construction is that, from the formulas for the variation of eigenvalue of the residue as a function of λ, a preferred section is always going to have some points where the eigenvalues are resonant, so we are not able just to impose a non-resonance condition on the residues.

We do however impose some conditions on the fiber over 0 so as to improve somewhat the moduli problem. This makes it so that the discussion will work for most but not all "preferred sections".

Framed quasi-parabolic logarithmic λ-connections

Let Y be a compact Riemann surface and $D \subset Y$ a reduced divisor, for simplicity we'll just suppose $D = \{y\}$. Set $X := Y - D$. Choose a base-point $x \in X$.

We look at bundles of rank 2.

Framed quasi-parabolic logarithmic λ-connections

A framed quasi-parabolic bundle with logarithmic λ-connection is

$$(\lambda, E, \nabla, F, \beta)$$

where $\lambda \in \mathbb{A}^1$, E is a rank 2 vector bundle on Y, ∇ is a logarithmic λ-connection i.e.

$$\nabla : E \to E \otimes \Omega^1_Y(\log D), \quad \nabla(ae) = a\nabla(e) + \lambda d(a)e$$

$F = F_y \subset E_y$ is a one-dimensional subspace preserved by $\mathrm{res}_y(\nabla)$, and $\beta : E_x \cong \mathbb{C}^2$ is a framing over the base-point.

Hypotheses

Those are the objects that our parameter space is going to parametrize. We make the following hypotheses $(*)$:

- The framed object is rigid: there are no automorphisms of (E, ∇) that preserve the framing β; and
- If $\lambda = 0$, the spectral curve of $\varphi = \nabla$ (which is a Higgs field in this case) is irreducible.
- If $\lambda \neq 0$ and the eigenvalues are resonant at all $y \in D$, the corresponding quasi-parabolic local system doesn't have all quasi-parabolic structures aligned on a single rank 1 sub-local system (roughly, there exists a stable parabolic weight system).

Moduli space

These conditions determine an open subspace of the moduli stack, and from the rigidity condition it is a space.
There exists a smooth algebraic space $\widetilde{M}_{\mathrm{Hod}}(X)$ locally of finite type, parametrizing the tuples $(\lambda, E, \nabla, F, \beta)$ satisfying Hypothesis $(*)$, together with a map

$$\widetilde{M}_{\mathrm{Hod}}(X) \xrightarrow{\lambda} \mathbb{A}^1.$$

Hecke-gauge groupoid

We define a *Hecke-gauge groupoid* $\mathscr{G}_{\mathrm{Hod}}$ acting on this moduli space. This groupoid consists of partially defined morphisms from the moduli space to itself, generated by the following operations on a tuple $(\lambda, E, \nabla, F, \beta)$

- (H): this is the Hecke operation going to the bundle $\ker(E \to E_y/F_y)$;
- (T): this is tensoring yielding $E \otimes \mathcal{O}_Y(y)$;
- (P): this operation is partially defined on the open set of points where the eigenvalues of $\mathrm{res}_y(\nabla)$ are distinct, yielding $(\lambda, E, \nabla, F^{\perp}, \beta)$ where $F_y^{\perp}$ is the eigenspace different from F_y.

Hecke-gauge groupoid

The Hecke-gauge groupoid is étale, that is to say the map

$$\mathscr{G}_{\mathrm{Hod}} \to \widetilde{M}_{\mathrm{Hod}}(X) \times \widetilde{M}_{\mathrm{Hod}}(X)$$

when composed with either of the projections is étale. This map is however not proper so the "quotient space", that would be denoted $M_{\mathrm{Hod}}(X)$, would be non-separated.

The quotient could be viewed as an analytic stack, but it doesn't have very good finiteness properties; it has particularly bad properties over $\lambda = 0$.

For this reason it will be better to just consider the groupoid as a whole.

Betti version

We can also define a "Betti" version of this moduli space. Let $\widetilde{M}_B(X)$ denote the moduli space of tuples (L, F, β) where L is a rank 2 local system on X, $F = \{F_y\}$ is a sub-local system of the restriction of L to a punctured disk around y, and $\beta : L_x \cong \mathbb{C}^2$ is a framing.

Let $\widetilde{M}_B(X)$ be the moduli space for such framed quasi-parabolic local systems.

Betti version of the Hecke-gauge groupoid

Define the *Betti Hecke-gauge groupoid* $\mathscr{G}_B$ acting on $\widetilde{M}_B(X)$ to be the groupoid consisting of partially defined morphisms from $\widetilde{M}_B(X)$ to itself, generated by the operation:

- (P): defined on the open set where the eigenvalues of the local monodromy transformation are distinct, sending (L, F, β) to $(L, F^{\perp}, \beta)$ where $F_y^{\perp}$ is the eigenspace of the local monodromy that is different from F_y.

Betti quotient space

We again get an etale groupoid. Here the finiteness conditions are better so the quotient $M_B(X) = \widetilde{M}_B(X)/\mathscr{G}_B$ is a non-separated algebraic stack.

Since a part of the Hecke-gauge groupoid comes from a partially-defined operation (P), the quotient stack has singular points that exhibit the "bug-eye" property observed by Janos Kollár (*J. Alg. Geom*, 1992).

Riemann-Hilbert

Theorem (Riemann-Hilbert correspondence)

We have an equivalence of analytic groupoids

$$\mathbb{G}_m \times_{\mathbb{P}^1} \left(\widetilde{M}_{\mathrm{Hod}}(X), \mathscr{G}_{\mathrm{Hod}}\right) \cong \mathbb{G}_m \times \left(\widetilde{M}_B(X), \mathscr{G}_B\right).$$

The isomorphism of topological spaces between X^{top} and $\overline{X}^{\mathrm{top}}$ gives an equivalence

$$\left(\widetilde{M}_B(X), \mathscr{G}_B\right) \cong \left(\widetilde{M}_B(\overline{X}), \mathscr{G}_B\right).$$

Deligne glueing

Using the Riemann-Hilbert correspondence we can then make a Deligne glueing. In terms of groupoids this can be viewed as follows: let

$$\widetilde{M}_{\mathrm{DH}} := \widetilde{M}_{\mathrm{Hod}}(X) \sqcup \widetilde{M}_{\mathrm{Hod}}(\overline{X})$$

with Hecke-gauge groupoid $\mathscr{G}_{\mathrm{DH}}$ combining the $\mathscr{G}_{\mathrm{Hod}}$ on both pieces, together with pieces identifying points that correspond to elements of the Betti moduli space that are identified under the previous equivalence:

$$M_{\mathrm{DH}} = \left(\widetilde{M}_{\mathrm{DH}}, \mathscr{G}_{\mathrm{DH}}\right).$$

Preferred section from a harmonic bundle

Suppose given a tame harmonic bundle $(E_X, \partial, \overline{\partial}, \varphi, \varphi^\dagger, h)$ on X, satisfying our condition that the spectral curve of the Higgs field is irreducible, and choose a framing $\beta : E_x \cong \mathbb{C}^2$.

For each $\lambda \in \mathbb{A}^1$, there is a well-defined extension of E_X to a parabolic bundle E^λ over Y (now including parabolic weights) together with a logarithmic λ-connection ∇_λ.

We recall that the parabolic structure reflects the growth properties of the harmonic metric.

Preferred section from a harmonic bundle

At our puncture y and for each λ, we get a pair of elements $(\mathfrak{p}_1(\lambda), \mathfrak{e}_1(\lambda))$ and $(\mathfrak{p}_2(\lambda), \mathfrak{e}_2(\lambda))$ in $\mathbb{R} \times \mathbb{C}$.

These all correspond to the same pair of elements of $\mathbb{R}^3 = \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))^\sigma$, modulo the action of the local Hecke-gauge group $\mathbb{Z}$.

We assume that the two elements are distinct modulo the gauge group—this is our non-resonance assumption.

Preferred section from a harmonic bundle

At general points λ, the parabolic weights will then be distinct, so the parabolic structure gives a well-defined quasi-parabolic structure, i.e. a rank 1 subspace $F_y \subset E_y$. This yields a point of $\widetilde{M}_{\mathrm{Hod}}$ well-defined up to $\mathscr{G}_{\mathrm{Hod}}$.

Notice that to get a bundle we need to pick one of the open intervals between parabolic weights; choosing a different interval is going to result in a change using the operations (H) and (T).

Preferred section from a harmonic bundle

There may, on the other hand, be some points at which the parabolic weights coincide in $\mathbb{R}/\mathbb{Z}$. Here, the parabolic structure no longer defines a quasi-parabolic structure, indeed the dimension jumps are by 2 at the unique parabolic weight in each interval $(k, k+1]$.

Our non-resonance assumption however implies, at these points, that the eigenvalues of $\mathrm{res}_y(\nabla)$ are different. Therefore, we may just choose one of the two eigenspaces to be F_y. Making the opposite choice yields a point that differs by the operation (P).

Thus, we also get a point of $\widetilde{M}_{\mathrm{Hod}}$ well-defined up to $\mathscr{G}_{\mathrm{Hod}}$.

Preferred section from a harmonic bundle

Putting these cases together, we see that for any $\lambda \in \mathbb{A}^1$ we obtain a point of $\widetilde{M}_{\mathrm{Hod}}$ well-defined up to $\mathscr{G}_{\mathrm{Hod}}$. One can check that locally these specifications patch together to give a section

$$\mathbb{A}^1 \to \left(\widetilde{M}_{\mathrm{Hod}}(X), \mathscr{G}_{\mathrm{Hod}}\right).$$

It patches with the same construction on the opposite chart to give a section of the Deligne-Hitchin space

$$\rho : \mathbb{P}^1 \to \left(\widetilde{M}_{\mathrm{DH}}(X), \mathscr{G}_{\mathrm{DH}}\right).$$

This is the *preferred section* corresponding to the harmonic bundle.

Mixed twistor normal bundle of the preferred section

The tangent bundle of the "quotient" $M_{\mathrm{DH}}(X)$ may be defined, since the groupoid is étale.

Theorem

Let $\mathcal{V}$ be the pullback by ρ of the tangent bundle of $M_{\mathrm{DH}}(X)$. It has a filtration

$$0 \subset W_0\mathcal{V} \subset W_1\mathcal{V} \subset W_2\mathcal{V} = \mathcal{V}$$

where $W_1\mathcal{V}$ is the set of tangent vectors that preserve the eigenvalues of the residues, and $W_0\mathcal{V}$ is the tangent space of the change of framing. Then $(\mathcal{V}, W.)$ is a mixed twistor structure, *meaning that W_k/W_{k-1} is a semistable bundle on $\mathbb{P}^1$ of slope k (for $k = 0, 1, 2$).*

Mixed twistor normal bundle of the preferred section

This theorem is the analogue in our case of the purity property in the compact case, and generalizes to rank 2 the weight 2 property observed for the rank 1 case in the previous paper.

Roughly speaking it should tell us that the moduli space of flat bundles with fixed local monodromy transformations has a hyperkähler structure. That is certainly already known in many cases, either by invoking the theory over compact orbicurves in the case of finite monodromy, or at least stated by physicists in a more general setting.

For the proof, we need a basic 1-dimensional part of the Sabbah-Mochizuki theory of mixed twistor $\mathscr{D}$-modules.

Happy Birthday Oscar!