

Newtonian gravitation as curvature of a space-time: Newton-Cartan gravitation and beyond.

What if the idea of connection had been defined in 1900?

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Old times with Oscar

AIP Journal of Mathematical Physics

Home > Journal of Mathematical Physics > Volume 29, Issue 5 > 10.1063/1.527946

Locally operating realizations of nonconnected transformation Lie groups

J. Math. Phys. **29**, 1083 (1988); https://doi.org/10.1063/1.527946

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What if the idea of a connection had been defined around 1900?

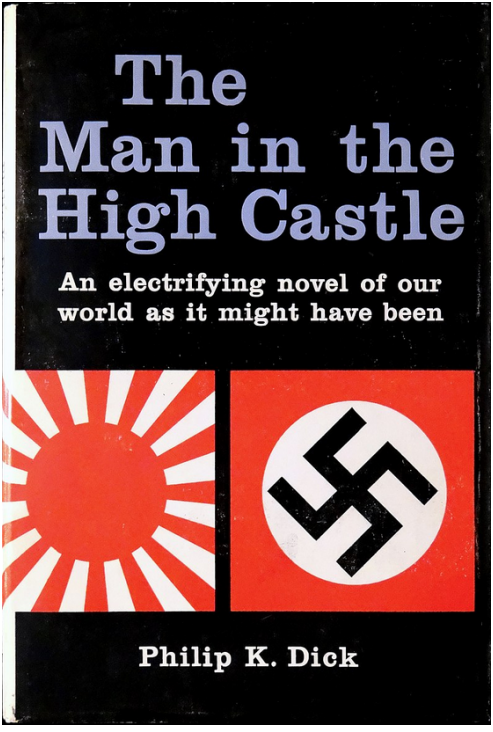
- In actual history, connections were defined by:
  - ⋆ Levi-Civita, (1917), 2D metric case, extrinsic
  - ⋆ Hermann Weyl, (1918), intrinsic reformulation
  - ⋆ Èlie Cartan, (1923 . . .), general idea of affine connection
- A postmature idea? What an actual historian of science has to say:

By the later third of the nineteenth century, all of the mathematical prerequisites for the introduction [of the concept of parallel displacement in Riemannian geometry and of a non-metric affine connection] were available. and it is an historical accident that they were not developed before the second decade of the twentieth century

This theory [of affine connections] allows a mathematically precise formulation of the Einstein's insight that gravitation and inertia are "essentially the same", as he put it. The absence of this context constituted a serious obstacle [....], indeed an unsurmountable one to its development by the mathematically most direct route. Consequently, Einstein was forced to take a detour through a long and indirect route [...] and to this day, many discussions of the interpretation of the general theory [...] still reflects the negative consequences of this detour.

▻ John Stachel, The story of Newstein or: Is gravity just anothe pretty force? in Jürgen Renn (ed.), The Genesis of General Relativity, Vol 4. 2006.

¿Uchronia?


The
Man in the
High Castle
An electrifying novel of our world as it might have been
Philip K. Dick


Philip
K. DICK
EL HOMBRE EN EL CASTILLO
CLÁSICOS MINOTAURO

Some quotations on uchronias in physics

... it is amusing and instructive to fantasyze how, in the best of all possible worlds, General Relativity might have been developed ab initio long before 1905, [....] and how it could then have led to Special Relativity.

Wolfgang Rindler, 1984

I believe that the construction of alternative historical scenarios [...], serves a valuable purpose in reminding us of the contingent and constructive nature of many features of a scientific 'discovery' that, with hindsight, we have become accustomed to regard as inevitable and natural.

John Stachel, 1994

Understanding the unfolding of any specific strecht of history greatly benefits from imagining how it could have developed otherwise, if such and such crucial circumstances had been different.

J.M. Lévy-Leblond, 2004

History as it developed

- ✓ 1670s - 1687 Newtonian Gravitation NTG
- ✓ 1861 - 1873 Electromagnetism
- ✓ 1905 *[Electromagnetism]* → Special Relativity
- ✓ 1915 *[Einstein vision]* → Einstein theory of gravitation, ETG
- ✓ 1917 Levi-Civita introduces parallel transport and connection
- ✓ 1923 Cartan shows Newtonian Gravitation as a geometric theory with a curved space-time

⋆ The absence of the idea of connection during the elaboration of ETG forced Einstein *"to take a detour through a long and indirect route [...] "*. Later, in the hands of Cartan, it was clear that Newtonian gravitation *could* be understood also through a curved space-time.

• Can we try to imagine how things could have evolved (with or without Einstein) if the idea of connection had been defined around 1900?

Outline

History as it was

- ⋆ Newtonian gravitation & inertial forces as could described in 1900 lenguaje
- ⋆ Special & General Relativity (1905 -1916)
- ⋆ Newtonian gravitation 'à la Cartan' (as actually developed starting 1923)

Now let us pretend connections had been defined in 1900: The uchronia starts

- ⋆ Uchronia from newtonian gravitation (starting say 1903)

Again to recent times:

- ⋆ The idea of a Classical Space-time

Presentation of science as it unfolded is not the only way to do it, nor even the best one. Alternative arguments and new derivations should be pursued and developed, not necessarily to replace, but at least to supplement the standard ones.

J.M. Lévy-Leblond, 2004

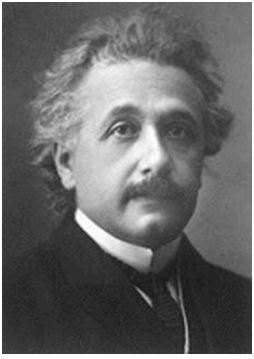
Dramatis personæ



K.F.Gauss



Bernhard Riemann



Albert Einstein



Isaac Newton



E.B.Christoffel



K. Schwarzschild

Karl A. Schwarzstein



T. Levi-Civita



Elie Cartan

Theme, Allegro

I. Newtonian gravitation & inertial forces, 1900

Newtonian Gravitation in 1900 (with some notational anachronisms)

- Actual concept of Inertial Frame had arisen 20 years before Lange, Neumann.
- The big mystery: Gravitational forces behave like Inertial forces.
  - ⋆ If a general RF moves with linear acceleration $w^i(t)$ and rotates with angular velocity given by $\Omega^i_j(t)$ (or its dual $\omega^k(t)$), the accelerations of a particle under a gravitational field in this RF are given by

$$m_{\text{inerc}}\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} = m_{\text{grav}}g^i(\mathbf{x},t) + m_{\text{inerc}}\left\{-w^i(t) + \left\{\dot{\Omega}^i_l - \Omega^i_j\Omega^j_l\right\}x^l + 2\Omega^i_j(t)\frac{\mathrm{d}x^j}{\mathrm{d}t}\right\}$$

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} = \frac{m_{\text{grav}}}{m_{\text{inerc}}}g^i(\mathbf{x},t) + \left\{-w^i(t) + \left\{\dot{\Omega}^i_l - \Omega^i_j\Omega^j_l\right\}x^l + 2\Omega^i_j(t)\frac{\mathrm{d}x^j}{\mathrm{d}t}\right\}.$$

  - ⋆ If $\frac{m_{\text{grav}}}{m_{\text{inerc}}} = 1$ universally, then

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} = g^i(\mathbf{x},t) + \left\{-w^i(t) + \left\{\dot{\Omega}^i_l - \Omega^i_j\Omega^j_l\right\}x^l + 2\Omega^i_j(t)\frac{\mathrm{d}x^j}{\mathrm{d}t}\right\}.$$

- This independence is the fundamental, profound property of gravity



H. Hertz

- Hertz was probably the first person to reopen the equality of inertial and gravitational masses as a fundamental problem .

And in reality we do have two properties before us, two most fundamental properties of matter, which must be thought of as being completely independent of each other, but in our experience, and only in our experience, appear to be exactly equal. This correspondence must mean much more than being just a miracle. . . . We must clearly realize, that the proportionality between mass and inertia must have a deeper explanation, and cannot be considered as of little importance, just as in the case of the equality of the velocites of electrical and optical waves. (Original in German, 1884.) *H. Hertz, Uber der Constitution der Materie, 1884.*

Can you separate/distinguish 'true gravity' from inertia?

- Leibniz, Clarke, Newton's bucket and the pirouette
  - ⋆ In Newton theory rotation is absolute
  - ⋆ Is also absolute the accelerated linear motion?
- To separate centrifugal and Coriolis effects of the rotation in the observed accelerations is possible:

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = g^i(\mathbf{x},t) - w^i(t) + \left\{\dot{\Omega}^i_l - \Omega^i_j\Omega^j_l\right\} x^l + 2\Omega^i_j(t)\frac{\mathrm{d}x^j}{\mathrm{d}t},$$

  - ⋆ But there is no any possible absolute separation of the 'true' gravitational effects from the inertial linear dragging

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = g^i(\mathbf{x},t) - w^i(t),$$

    - ✓ Apparent cancelation of g by w can happen just at one point, but not in a small region (tidal effects).
- This idea should be called Newtonian Equivalence principle: It should hold if $\frac{m_{\text{grav}}}{m_{\text{inerc}}} = 1$ universally

CounterTheme, Allegro

Ib A Glimpse of Einstein's Relativities

Einstein Special Relativity and Einstein theory of gravity

- ⋆ SR describes space-time in absence of gravity (an idealisation)
- ⋆ ETG describes space-time in presence of gravity

• Space-time is a pseudo-riemannian space with a lorentzian metric $g_{\alpha\beta}$ Signature $(+---)$

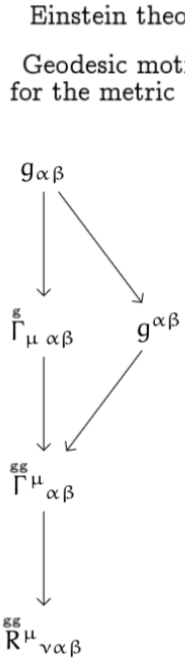
- ⋆ Lorentzian signature metrics . . . only after Grossmann/Einstein 1913.
  - ✓ Implicit in the context of projective geometry by Cayley and Klein.

• Geodesic Hypothesis Free fall follows the geodesics of this metric.

• Curvature. Curvature tensor is determined by the presence of energy/momentum through Einstein's equations (plus the differential Bianchi relations which are automatic identities). Special relativity ignores gravitation, hence the corresponding curvature tensor vanishes

- ⋆ Without Einstein, Special Relativity would have been likely proposed by some other author a short time later
- ⋆ General Relativity is different: it was an Einstein's personal and obstinate creation

The three levels in Einstein's theory



✓ Metric tensor $\equiv$ { Chronogeometry of space-time; Gravitational potentials

✓ Levi-Civita connection $\equiv$ { Intensity of the Gravitational field; Intensity of the Gravimagnetic field

✓ Riemann curvature tensor $\equiv$ { Tidal force field; Tidal Gravimagnetic field; 3-Spatial curvature

Einstein on Newton, absolute space and inertia

One can see from the way he [Newton] formulated his views that Newton felt by no means comfortable about the concept of absolute space, which embodied that of absolute rest; for he was alive to the fact that nothing in experience seemed to correspond to this latter concept. He also felt uneasy about the introduction of action at a distance. But the enormous practical success of his theory may well have prevented him and the physicists of the eighteenth and nineteenth centuries from recognising the fictitious [i.e. creative] character of the principles of his system.

A. Einstein 1934, Herbert Spencer lecture "On the Method of Theoretical Physics"

Can gravitation and inertia be identical? This question leads directly to the General Theory of Relativity. Is it not possible for me to regard the earth as free from rotation, if I conceive of the centrifugal force, which acts on all bodies at rest relatively to the earth, as being a "real" gravitational field of gravitation, or part of such a field? [...] My conviction of the identity of inertial and gravitational mass aroused within me the feeling of absolute confidence in the correctness of this interpretation.

A. Einstein 1921. A Brief Outline . . . , Nature,.

- Basic idea: Gravitational forces and inertial forces should be aspects of a single phenomenon . . . [And now we can add: "to be described through a single connection"].

Main variation, Andante

II. Newtonian gravitation 'à la Cartan' (1923 - . . .)

Can newtonian gravitation be formulated in a fully covariant way?

Although it is true that every empirical law can be put in a generally covariant form, yet the principle of relativity possesses a great heuristic power....Of two theoretical systems, both of which agree with experience, the one is to be preferred which, from the point of view of the absolute differential calculus is the simpler and more transparent. Let one express Newtonian mechanics four-dimensionally in the form of generally covariant equations and one will surely be convinced that the principle of relativity excludes this theory from the practical, though not the theoretical, viewpoint.

A. Einstein, 1918, response to the E. Kretschmann 1917 criticism.

Covariant formulations of Newton's theory of gravitation proposed by Élie Cartan (1923, 1924) y K. Friedrichs (1927) dashed the expectations Einstein had taken for granted. They are not less simple, yet, I would say, more transparent than the original version by Newton (original in Spanish)

Roberto Torretti, 1994, La geometría del Universo y otros ensayos

Structures in Newton-Cartan Space-time: I Chronogeometry

- Chronogeometrical structures: Temporal durations and spatial distances
- Described by two tensors
  - ✓ In adapted coordinates $x^\mu \equiv (t, x^i)$, with t absolute time, and the three x^i cartesian coordinates in each simultaneity 3-space

$$t_\mu = \left(\begin{array}{c} 1 \\ \hline 0 \\ 0 \\ 0 \end{array}\right) \qquad t_{\mu\nu} = t_\mu t_\nu = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad h^{\mu\nu} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

  - ✓ in all space-time
  - ✓ invariant under rigid motions and satisfying a natural compatibility condition

$$t_\mu h^{\mu\alpha} = 0, \qquad t_{\mu\nu} h^{\nu\alpha} = 0$$

- These two tensors are absolute, not dynamical
  - ⋆ Ideal clocks register (the 'temporal length of') curves measured after $t_{\mu\nu}$, ideal rulers measure a "spatial length" associated to $h^{\mu\nu}$. Gravitational field does not affect them

Structures in Newtonian space-time: II Inertia law

- Gravitation + inertia should be described though a connection
- What is the actual connection describing free fall?

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} = 0 \qquad \frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} + \Gamma^i{}_{00} + 2\Gamma^i{}_{j0}\frac{\mathrm{d}x^j}{\mathrm{d}\tau} + \Gamma^i{}_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}\frac{\mathrm{d}x^k}{\mathrm{d}\tau} = 0, \qquad i = 1, 2, 3,$$

From the first equation $\tau = t$. Substituting in the second, working in an inertial frame and comparing with

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = g^i(\mathbf{x}, t)$$

the components of the connection are given in a general reference frame in terms of the gravitational field $g^i(\mathbf{x}, t)$ by:

$$\Gamma^0{}_{\alpha\beta} = 0$$
$$\Gamma^i{}_{00} = -g^i(\mathbf{x}, t) \qquad \Gamma^i{}_{0j} = \Gamma^i{}_{j0} = 0 \qquad \Gamma^i{}_{jk} = 0$$

Using adapted coordinates in a non-inertial reference frame, with non-inertiality data $w^i(t)$ and $\Omega^i{}_j(t)$, the connection has components:

$$\Gamma^0{}_{\alpha\beta} = 0$$
$$\Gamma^i{}_{00} = -g^i(\mathbf{x}, t) + w^i(t) - \left\{\dot{\Omega}^i_l - \Omega^i_j\Omega^j_l\right\} x^l \quad \Gamma^i{}_{0j} = \Gamma^i{}_{j0} = -\Omega^i_j(t) \quad \Gamma^i{}_{jk} = 0$$

Compatibility between Chronogeometry and free fall connection

- A compatibility condition of the connection with temporal metric $t_{\mu\nu}$ and space 'cometric' $h^{\mu\nu}$
  - ✓ Compatibility conditions are (in general coordinates)

$$t_\mu \Gamma^\mu{}_{\alpha\beta} = 0, \qquad \Gamma^k{}_{\lambda\alpha} h^{\lambda l} + \Gamma^l{}_{\lambda\alpha} h^{\lambda k} = 0$$

- ⋆ Connection is a new object, additional to the chronogeometric structure
  - ✓ Warning: a compatible connection with the chronogeometric structure $t_{\mu\nu}$ and $h^{\mu\nu}$ is not unique: there are infinite such connections
  - ✓ Inertial connection $\widetilde{\Gamma}^\mu{}_{\alpha\beta}$ and Free fall connection $\Gamma^\mu{}_{\alpha\beta}$ are two of these

$$\widetilde{\Gamma}^0{}_{\alpha\beta} = 0$$

$$\widetilde{\Gamma}^i{}_{00} = \qquad w^i(t) - \left\{ \dot{\Omega}^i_l - \Omega^i_j \Omega^j_l \right\} x^l \quad \widetilde{\Gamma}^i{}_{0j} = \widetilde{\Gamma}^i{}_{j0} = -\Omega^i_j(t) \quad \widetilde{\Gamma}^i{}_{jk} = 0$$

$$\Gamma^0{}_{\alpha\beta} = 0$$

$$\Gamma^i{}_{00} = -g^i(\mathbf{x}, t) + w^i(t) - \left\{ \dot{\Omega}^i_l - \Omega^i_j \Omega^j_l \right\} x^l \quad \Gamma^i{}_{0j} = \Gamma^i{}_{j0} = -\Omega^i_j(t) \quad \Gamma^i{}_{jk} = 0$$

Curvature tensor of the inertial and of free-fall connection

- Let $\widetilde{R}^{\mu}{}_{\nu\alpha\beta}$ and $R^{\mu}{}_{\nu\alpha\beta}$ be the curvature tensors of the inertial $\widetilde{\Gamma}^{\mu}{}_{\alpha\beta}$ and of the free-fall connection $\Gamma^{\mu}{}_{\alpha\beta}$
  - ⋆ The inertial connection $\widetilde{\Gamma}^{\mu}{}_{\alpha\beta}$ is flat, gravity is an external field $\mathbf{g} = -\boldsymbol{\nabla}\varphi$
    - ✓ Equations of newtonian gravitational field

$$\boldsymbol{\nabla}\cdot\mathbf{g} = -4\pi G\rho_m, \qquad \boldsymbol{\nabla}\times\mathbf{g} = 0 \quad \Longleftrightarrow \quad 4\pi G\rho_m = \nabla^2\varphi$$

  - ⋆ The free-fall connection $\Gamma^{\mu}{}_{\alpha\beta}$ has non-zero curvature

$$R^{i}{}_{0m0} = -\frac{\partial\, g^{i}}{\partial\, x^{m}} = \delta^{il}\frac{\partial^2\varphi}{\partial\, x^{l}\partial\, x^{m}}, \quad R^{i}{}_{00m} = -R^{i}{}_{0m0}, \quad \text{remaining } R^{\lambda}{}_{\nu\alpha\beta} = 0$$

    - ✓ $R^{i}{}_{0m0}$ identified with the newtonian tidal field, this is, the 'gradients' of the ordinary intensity of gravitational field
    - ✓ But this curvature is only 'temporal'; the 3-space is flat
    - ✓ Ricci tensor

$$R_{\nu\beta} := R^{\lambda}{}_{\nu\lambda\beta}, \quad R_{00} := R^{\lambda}{}_{0\lambda0} = h^{ml}\frac{\partial^2\varphi}{\partial\, x^{l}\partial\, x^{m}} = \boldsymbol{\nabla}\varphi \quad \text{remaining } R_{\nu\beta} = 0$$

    - ✓ Equations of newtonian gravitational field

$$\boldsymbol{\nabla}\cdot\mathbf{g} = -4\pi G\rho_m, \qquad \boldsymbol{\nabla}\times\mathbf{g} = 0 \quad \Longleftrightarrow \quad R_{00} := 4\pi G\rho_m$$

Newtonian theory as seen before / as could be seen after Cartan

- Before: Just a single view
  - ⋆ Conventional view, absolute time, euclidean 3-space, flat space-time, gravity is an external force
- After: Two possible interpretations:
  - ⋆ Geometrized Conventional view: absolute time, euclidean 3-space, flat space-time, inertial forces are necessarily described by a (flat) connection which supersedes absolute space, gravity is still an external force
  - ⋆ Fully Geometrized Newton-Cartan version: absolute time remains, absolute space replaced by a connection which describes jointly gravitation and inertia; this connection has curvature, but 3-space remains flat, gravity is a manifestation of curvature in space-time
    - ▷ Cartan 1923, Friedrichs 1927, Havas 1964, Trautman 1965, ...
- Summing up: two alternative possible interpretations, both with absolute time
- Deep common element in ETG and in both approaches to NGT: Further to the chronogeometric structure, Galilei principle and inertia principle **require a connection in space-time.**
  - ⋆ But Newton-Cartan one is preferable, as inertial connection is unobservable

Teachers confronted to a difficulty

In this spirit, we devote this article to a theme which is mainly of conceptual nature, and – as we hope – also of some pedagogical interest [...]

Every honest teacher of theoretical physics is confronted at a very early stage of a classical mechanics course with the following difficulty: After having introduced —in the spirit of L. Lange— the operational definition of an inertial frame, the question arises how to proceed when gravitational fields are present. In the traditional presentation of Newton's theory one maintains the fiction of an integrable (flat) affine connection, and puts gravity on the side of the forces, described by vector fields. A much more satisfactory formulation was given by Cartan and Friedrichs. This denies the separate existence of a flat affine connection of space-time and a vector field describing gravitation, but puts gravity on the side of a more general dynamical connection which represents both inertia and gravitation.

Historically, this important step of course was made first by Einstein when he created his general theory of relativity, but it is clearly independent of the relativization of time. Following Cartan and Friedrichs, numerous authors have elaborated on this idea.

C. Rüede and N. Straumann, "On Newton-Cartan Cosmology", 1996

Variations, Ad libitum

III. An uchronia: (starting say 1903)

Lagrangian formulation of gravitation

- In a inertial RF

$$L = \tfrac{1}{2} m \delta_{IJ} \dot{x}^I \dot{x}^J - m\varphi(t, \mathbf{x})$$

$$\frac{\mathrm{d}^2 x^I}{\mathrm{d}t^2} = g^I(t, \mathbf{x}), \qquad g^I(t, \mathbf{x}) := -\delta^{IJ} \frac{\partial\, \varphi(t, \mathbf{x})}{\partial\, x^J}$$

- In a non-inertial RF, how about the lagrangian? Requiring L is a scalar in the coordinate change from the inertial to the noninertial RF

$$L = \tfrac{1}{2} m \delta_{ij} \dot{x}^i \dot{x}^j - m\varphi(t, \mathbf{x}) - m\delta_{ij} w^i x^j + \tfrac{1}{2} m \delta_{ij} \Omega^i{}_k \Omega^j{}_l x^k x^l - m\delta_{ij} \Omega^i{}_k x^k \dot{x}^j + m \frac{d}{dt}(\ldots)$$

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = g^i(t, \mathbf{x}) - w^i(t) + \left\{ \dot{\Omega}^i_l - \Omega^i_j \Omega^j_l \right\} x^l + 2\Omega^i_j(t) \frac{\mathrm{d}x^j}{\mathrm{d}t},$$

  - ⋆ All forces appear through a 'velocity dependent potential' Φ
    - ✓ Let us call this gravitokinetic potential, and group after the degree in the velocity $\mathbf{v} \equiv \dot{\mathbf{x}}$:

$$m\Phi = m\varphi(t, \mathbf{x}) + m\delta_{ij} w^i x^j - \tfrac{1}{2} m \delta_{ij} \Omega^i{}_k \Omega^j{}_l x^k x^l + m\delta_{ij} \Omega^i{}_k x^k \dot{x}^j$$

$$L = \tfrac{1}{2} m \delta_{ij} v^i v^j - m\big(\mathcal{U} - \mathcal{V}_j v^j\big) = \tfrac{1}{2} m \mathbf{v}^2 - m\big(\mathcal{U} - \boldsymbol{\mathcal{V}} \cdot \mathbf{v}\big)$$



K.Schwarzschild

- In 1903 K. Schwarzschild found the correct Lagrangian for a charged particle in an external electromagnetic field. The complete potential, with linear dependence on the velocities, was called Electrokinetic potential by Schwarzschild
  - ⋆ Quite similar arguments could lead to the Lagrangian for a mass under gravitational and inertial forces and these might have been put forward by K. Schwarzschild

How this gravitokinetic potential transforms

- From the known potentials $(\mathcal{U}, \boldsymbol{\mathcal{V}})$ in a reference frame, can you get these potentials in another reference frame obtained through a rigid motion)?
- Essential step: work in the 4D newtonian space-time (in adapted coordinates)

$$\text{(Newtonian) fourvelocity } u^k = \frac{\mathrm{d}x^k}{\mathrm{d}\tau} = \frac{\mathrm{d}x^k}{\mathrm{d}t}, \qquad u^\mu \equiv \begin{pmatrix} 1 \\ v^k \end{pmatrix},$$

✓ Lagrangian in this four-dimensional lenguaje

$$-m\,\mathcal{U} + m\,\mathcal{V}_k \frac{u^k}{u^0} + \tfrac{1}{2} m\,\delta_{ij} \frac{u^i}{u^0}\frac{u^j}{u^0} = \frac{m}{2(u^0)^2}\Big(-2\,\mathcal{U}\,(u^0)^2 + 2\,\mathcal{V}_k u^k u^0 + \delta_{ij} u^i u^j \Big)$$

✓ is a quadratic form in the fourvelocity

$$L = l_{\mu\nu} u^\mu u^\nu = l_{00}(u^0)^2 + l_{i0} u^i u^0 + l_{0j} u^0 u^j + l_{ij} u^i u^j$$

- L will be an scalar iff the coefficients $l_{\mu\nu}$ de $l_{\mu\nu} u^\mu u^\nu$ are a (0,2) type tensor
- This tensor fully encodes the dynamic, and it does irrespective of the inertial or noninertial nature of the RF . This is essential if the inertial or non inertial character of the frame cannot be experimentally ascertained.



H. von Seeliger



K.Schwarzschild

- Hugo von Seeliger, the Ph.D supervisor of K. Schwarzschild, had tried in the 1890s to resolve the old Newton riddle about the behaviour of a particle under the gravitation of an infinite mass distribution with uniform density
  - ⋆ This prompted Schwarzschild in 1897, just after he got his Ph.D.,to establish some links between gravitation and inertia, and almost to conclude that to be an inertial system has no an independent status but is a partially conventional idea

How this gravitokinetic potential transforms II

- Free fall dynamics in a gravitational field, seen from a general RF, follows the extremals of the action associated to the lagrangian $L = l_{\mu\nu} u^\mu u^\nu$
  - ⋆ In a non rotating system, which can be considered as (conventionally) inertial (gravitational potential $\varphi + \mathbf{w}(t)\cdot\mathbf{x}$):

$$l_{\mu\nu} = \begin{pmatrix} -2\mathcal{U} & \mathcal{V}_j \\ \mathcal{V}_i & \delta_{ij} \end{pmatrix} = \begin{pmatrix} -2\varphi - 2\mathbf{w}(t)\cdot\mathbf{x} & \mathbf{0} \\ \mathbf{0} & \delta_{ij} \end{pmatrix}$$

  - ⋆ From a general, rotating reference frame

$$l_{\mu\nu} = \begin{pmatrix} -2\mathcal{U} & \mathcal{V}_j \\ \mathcal{V}_i & \delta_{ij} \end{pmatrix} = \begin{pmatrix} -2\varphi - 2\mathbf{w}(t)\cdot\mathbf{x} + (\boldsymbol{\omega}(t)\times\mathbf{x})^2 & (\boldsymbol{\omega}(t)\times\mathbf{x})_j \\ (\boldsymbol{\omega}(t)\times\mathbf{x})_i & \delta_{ij} \end{pmatrix}$$

- Transformation law of the inertial potentials follows automatically from the tensor character of $l_{\mu\nu}$. The inhomogeneity of the transformation law (inertial potentials might vanish in a RF but not in other) does not conflict with the purely linear tensor transformation law as $l_{\mu\nu}$ is a tensor with constant, nonzero space-space elements



Ll. Bel

- In fact, symmetric tensors of valence (0,2) with space-space components δ_{ij} preserve this condition under any rigid transformation
- Amazingly, this particular property of these tensors has only been pointed out in the 1990s by Lluis Bel

Connection enters stage

- Feign connection had been defined around 1900
- Then, the way was paved to describe the full set 'gravitation & inertia' by a connection in four dimensional space-time
  - ⋆ Autoparallels of a connection in (newtonian) space-time are worldlines of particles $\tau \to x^\mu(\tau)$ whose tangent vector $\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}$ is in parallel transport

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu{}_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0$$

- This fits (and 'explain') the Galileo principle because free fall dos not depend on any property of the falling particle
  - ⋆ Several possibilities for the connection?
    - ✓ A 'purely inertial connection' describing only inertial forces yet keeping gravitational field as an external force.
    - ✓ A 'free fall connection' describing jointly inertial and gravitational forces
  - ⋆ Newtonian equivalence principle implies that the purely inertial connection is unobservable
- A single free fall connection should describe gravitational & inertial forces

What is the actual free fall connection?

- Take the conventional free fall equation in the form (in adapted coordinates)

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} = g^i(\mathbf{x},t) - w^i(t) + \left\{\dot{\Omega}^i_l - \Omega^i{}_j\Omega^j_l\right\}x^l + 2\Omega^i_j(t)\frac{\mathrm{d}x^j}{\mathrm{d}t},$$

  - ⋆ Rewrite as

$$\frac{\mathrm{d}^2t}{\mathrm{d}\tau^2} = 0 \qquad \frac{\mathrm{d}^2x^i}{\mathrm{d}\tau^2} + \Gamma^i{}_{00} + 2\Gamma^i{}_{j0}\frac{\mathrm{d}x^j}{\mathrm{d}\tau} + \Gamma^i{}_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}\frac{\mathrm{d}x^k}{\mathrm{d}\tau} = 0, \qquad i = 1,2,3,$$

  - ⋆ Eliminate τ replacing in the spatial part and compare with

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} = \left(g^i(\mathbf{x},t) - w^i(t) + \left\{\dot{\Omega}^i_l - \Omega^i{}_j\Omega^j_l\right\}x^l\right) + 2\Omega^i_j(t)\frac{\mathrm{d}x^j}{\mathrm{d}t},$$

  - ⋆ Then read directly the components of the free fall connection as expressed in terms of the gravitational field $g^i(\mathbf{x},t)$ and the non inertiality data of the reference frame, $w^i(t)$ y $\Omega^i{}_j(t)$:

$$\Gamma^0{}_{\alpha\beta} = 0 \qquad \begin{aligned} \Gamma^i{}_{00} &= -g^i(\mathbf{x},t) + w^i(t) - \left\{\dot{\Omega}^i_l - \Omega^i{}_j\Omega^j_l\right\}x^l \\ \Gamma^i{}_{0j} &= \Gamma^i{}_{j0} = -\Omega^i_j(t) \\ \Gamma^i{}_{jk} &= 0 \end{aligned}$$

Chronogeometry in the newtonian space-time

- Temporal durations and spatial distances Geometric properties, with meaning independent of any coordinates used.
  - ⋆ Invariant under rigid motions in 3-spaces of simultaneity: described by tensors
    - ✓ In adapted coordinates, $x^\mu \equiv (t, x^j)$ the transformation law under reference frames changes is:

$$t' = t, \qquad x^{i'} = R^{i'}{}_j(t)\,\left(x^j - a^j(t)\right)$$

- Chrono-geometricstructure described by two tensors

$$t_\mu = \left(\begin{array}{c|ccc} 1 & & & \\ \hline 0 & & & \\ 0 & & & \\ 0 & & & \end{array}\right) \qquad t_{\mu\nu} = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad h^{\mu\nu} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

  - ✓ holding in all newtonian space-time, invariant under rigid motions
  - ✓ Compatibility

$$t_\mu h^{\mu\alpha} = 0, \qquad t_{\mu\nu} h^{\nu\alpha} = 0$$

- The tensors $t_{\mu\nu}$, are absolute, not dynamic. Gravitational field does not affect them Ideal cloks measure the (temporal lenght) of $t_{\mu\nu}$ taken as a metric.

Joining the dots: Free fall connection in terms of $l_{\mu\nu}$ and $h^{\mu\nu}$

- A direct checking shows that the free fall connection is uniquely determined by the two tensors $l_{\mu\nu}$ and $h^{\mu\nu}$ as

$$\Gamma^{\mu}{}_{\alpha\beta} = h^{\mu\lambda}\tfrac{1}{2}\left(-\frac{\partial l_{\alpha\beta}}{\partial x^{\lambda}} + \frac{\partial l_{\lambda\beta}}{\partial x^{\alpha}} + \frac{\partial l_{\alpha\lambda}}{\partial x^{\beta}}\right)$$

$$\Gamma^{0}{}_{\alpha\beta} = 0$$

$$\Gamma^{i}{}_{00} = \delta^{il}\tfrac{1}{2}\left(-\frac{\partial l_{00}}{\partial x^{l}} + \frac{\partial l_{l0}}{\partial x^{0}} + \frac{\partial l_{0l}}{\partial x^{0}}\right) = \delta^{il}\left(\frac{\partial \mathcal{U}}{\partial x^{l}} + \frac{\partial \mathcal{V}_l}{\partial t}\right)$$

$$\Gamma^{i}{}_{j0} = \Gamma^{i}{}_{0j} = \delta^{il}\tfrac{1}{2}\left(-\frac{\partial l_{0j}}{\partial x^{l}} + \frac{\partial l_{0l}}{\partial x^{j}}\right) = -\delta^{il}\tfrac{1}{2}\left(\frac{\partial \mathcal{V}_j}{\partial x^{l}} - \frac{\partial \mathcal{V}_l}{\partial x^{j}}\right),$$

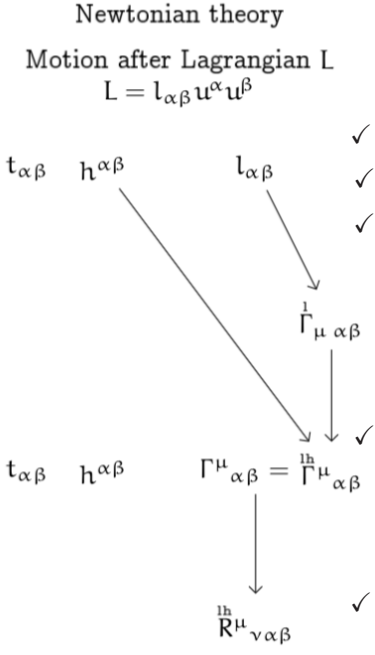
$$\Gamma^{i}{}_{jk} = 0$$

  - ⋆ and this completely fits with the equation of the extremals of the action with Lagrangian L

$$L = \tfrac{1}{2}m\delta_{ij}v^{i}v^{j} - m\big(\mathcal{U} - \mathcal{V}_j v^{j}\big) = \tfrac{1}{2}m\mathbf{v}^{2} - m\big(\mathcal{U} - \boldsymbol{\mathcal{V}}\cdot\mathbf{v}\big)$$

$$\frac{\mathrm{d}^{2}\mathbf{x}^{k}(t)}{\mathrm{d}t^{2}} = \delta^{kl}\left(-\frac{\partial \mathcal{U}}{\partial x^{l}} - \frac{\partial \mathcal{V}_l}{\partial t}\right) + \delta^{kl}\left(\frac{\partial \mathcal{V}_j}{\partial x^{l}} - \frac{\partial \mathcal{V}_l}{\partial x^{j}}\right)\frac{\mathrm{d}\mathbf{x}^{j}(t)}{\mathrm{d}t},$$

The three levels in this theory



✓ Temporal metric tensor $t_{\alpha\beta} \equiv$ Space-time Chronometry

✓ Spatial cometric tensor $h^{\alpha\beta} \equiv$ Geometry of 3D space

✓ Lagrangian Tensor $l_{\alpha\beta} \equiv$ Gravitational potentials

✓ Connection $\equiv$ Intensity of the Gravitational field

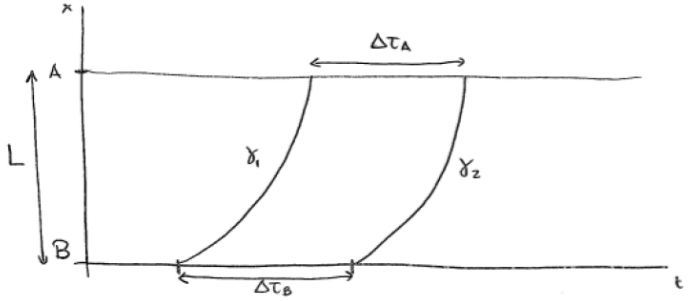
✓ Riemann curvature tensor $\equiv$
- Tidal forces
- NO Gravimagnetic field
- NO Spatial curvature

Geodesics of a metric also?

- So far, and depending on the context, free fall can be seen
  - ⋆ In a Lagrangian context: extremals of an action
  - ⋆ In a 'ProtoCartan' context, autoparallels of a connection.
- In a general RF, free fall follows the extremals of the variational problem with action functional $\int l_{\mu\nu} u^\mu u^\nu dt$, which coincide with the autoparallels of the connection $\Gamma^\mu{}_{\alpha\beta}$.
- Now a natural question would be to try a third interpretation:
  - ⋆ Free fall as Geodesics of a metric?
- This is our last step.
  - ✓ As action is an integral along time this suggests to look for a 'metric' whose 'length' along curves be the time.
  - ⋆ The only candidate we already have is the temporal degenerate metric tensor $t_{\mu\nu}$. But the degeneracy implies that the 'time length' of a curve, as computed with this 'metric tensor' is path independent (and equals the absolute time lapse between starting and ending events).

Is absolute time observable in a gravitational field?

- Einstein-Schild argument



⋆ Light with frequency ν_B emitted at B and received at A, at height L above B in a uniform gravitational field g. What is the received frequency ν_A ?

⋆ Newtonian equivalence principle implies unavoidably a frequency shift

✓ If light is taken as a real clock, (through its frequency, not through its propagation in space) this means that real clocks cannot measure absolute time

⋆ So what is the tensor giving the really measured time as the 'temporal length' along its worldline?



A. Einstein

- This idea was actually advanced for Einstein in 1907 and then in 1911, and was based exclusively in newtonian physics

How can the actual 'observable time' be described ?

- Basic hypothesis: Actually measured durations are not described by the old 'absolute time tensor $t_{\mu\nu}$ but by another one $n_{\mu\nu}$
  - ⋆ Then absolute time is now considered unobservable
    - ▻ (alike the purely inertial connection)
- For $n_{\mu\nu}$,considerable quantitative evidence in favour of absolute time suggest that $n_{\mu\nu}$ is only a small correction to $t_{\mu\nu}$. As $t_{\mu\nu}$ and $l_{\mu\nu}$ are tensors of the same valence, the simplest candidate for $n_{\mu\nu}$ could be

$$n_{\mu\nu} := t_{\mu\nu} + \kappa\, l_{\mu\nu}, \qquad \text{dimensionally } \kappa \text{ is } \mathsf{L}^{-2}\mathsf{T}^2 \qquad \kappa = -\tfrac{1}{c^2}$$

- At the antropocentric scale, the speed c must be very large Maxwell.
  - ⋆ The case of light climbing an uniform gravitational field even suggests that c must coincide with the speed of light.

$$l_{\alpha\beta} = \begin{pmatrix} -2\mathcal{U} & \mathcal{V}_j \\ \mathcal{V}_i & \delta_{ij} \end{pmatrix} \qquad n_{\alpha\beta} := t_{\alpha\beta} - \tfrac{1}{c^2} l_{\alpha\beta} = \begin{pmatrix} 1 + \frac{2\mathcal{U}}{c^2} & -\frac{\mathcal{V}_x}{c^2} & -\frac{\mathcal{V}_y}{c^2} & -\frac{\mathcal{V}_z}{c^2} \\ -\frac{\mathcal{V}_x}{c^2} & -\frac{1}{c^2} & 0 & 0 \\ -\frac{\mathcal{V}_y}{c^2} & 0 & -\frac{1}{c^2} & 0 \\ -\frac{\mathcal{V}_z}{c^2} & 0 & 0 & -\frac{1}{c^2} \end{pmatrix}$$

Is free fall a geodesic of this non-degenerate metric $n_{\mu\nu}$? Not quite ...

- Now $n_{\mu\nu}$ can be considered a metric, which turns out to be non degenerate. Its metric connection is:

$$\overset{n}{\Gamma}{}^{\mu}{}_{\alpha\beta} = n^{\mu\lambda}\overset{n}{\Gamma}_{\lambda\,\alpha\beta}, \qquad \overset{n}{\Gamma}_{\lambda\,\alpha\beta} = \tfrac{1}{2}\left(-\frac{\partial n_{\alpha\beta}}{\partial x^{\lambda}} + \frac{\partial n_{\lambda\beta}}{\partial x^{\alpha}} + \frac{\partial n_{\alpha\lambda}}{\partial x^{\beta}}\right)$$

  - ⋆ A series in powers of $1/c^2$

- If now we develop in powers of $1/c^2$, the newtonian free fall connection is directly the leading order 0 term:

$$\begin{aligned}
\overset{n}{\Gamma}{}^{0}{}_{00} &= n^{0\mu}\overset{n}{\Gamma}_{\mu\,00} = && 0 \quad + \cdots \\
\overset{n}{\Gamma}{}^{i}{}_{00} &= n^{i\mu}\overset{n}{\Gamma}_{\mu\,00} = \delta^{il}\tfrac{1}{2}\left(-\frac{\partial l_{00}}{\partial x^{l}} + \frac{\partial l_{l0}}{\partial x^{0}} + \frac{\partial l_{0l}}{\partial x^{0}}\right) + \cdots \\
\overset{n}{\Gamma}{}^{0}{}_{j0} = \overset{n}{\Gamma}{}^{0}{}_{0j} &= n^{0\mu}\overset{n}{\Gamma}_{\mu\,j0} = && 0 \quad + \cdots \\
\overset{n}{\Gamma}{}^{i}{}_{j0} = \overset{n}{\Gamma}{}^{i}{}_{0j} &= n^{i\mu}\overset{n}{\Gamma}_{\mu\,j0} = \delta^{il}\tfrac{1}{2}\left(-\frac{\partial l_{0j}}{\partial x^{l}} + \frac{\partial l_{0l}}{\partial x^{j}}\right) + \cdots \\
\overset{n}{\Gamma}{}^{0}{}_{jk} &= n^{0\mu}\overset{n}{\Gamma}_{\mu\,jk} = && 0 \quad + \cdots \\
\overset{n}{\Gamma}{}^{i}{}_{jk} &= n^{i\mu}\overset{n}{\Gamma}_{\mu\,jk} = && 0 \quad + \cdots
\end{aligned}$$

From the tensor $n_{\alpha\beta}$ to the old newtonian gravitation

- The newtonian free fall connection $\Gamma^{\mu}{}_{\alpha\beta}$ *is not the metric connection of the tensor* $n_{\mu\nu}$ seen as a metric, but coincides exactly with the leading terms of order 0 in $1/c^2$ of this metric connection
  - ✓ Whithin the theory based on $n_{\mu\nu}$, actual free fall is not exactly described by the putative newtonian free fall
- The complete geometric structure of the newtonian gravitation follows as a limit of the new theory
  - ⋆ Temporal metric $t_{\mu\nu}$ (degenerate, not dynamical) is the limit $c \to \infty$ of the tensor $n_{\mu\nu}$
  - ⋆ Spatial co metric $h^{\mu\nu}$ (degenerate, not dynamical) is the limit $c \to \infty$ of the tensor $(\frac{-1}{c^2})n^{\mu\nu}$
  - ⋆ Newtonian free fall connection $\Gamma^{\mu}{}_{\alpha\beta}$ (ddynamical) is the limit $c \to \infty$ of the connection $\Gamma^{\overset{n}{\mu}}{}_{\alpha\beta}$
- All this clearly suggest newtonian gravitation **might be only an approximate theory**

What is new in the $n_{\mu\nu}$ tensor?

- General form of $n_{\alpha\beta}$ when there are a gravitational field and inertial forces

$$n_{\alpha\beta} := t_{\alpha\beta} - \tfrac{1}{c^2} l_{\alpha\beta} = \begin{pmatrix} 1 + \frac{2\mathcal{U}}{c^2} & -\frac{\mathcal{V}_x}{c^2} & -\frac{\mathcal{V}_y}{c^2} & -\frac{\mathcal{V}_z}{c^2} \\ -\frac{\mathcal{V}_x}{c^2} & -\frac{1}{c^2} & 0 & 0 \\ -\frac{\mathcal{V}_y}{c^2} & 0 & -\frac{1}{c^2} & 0 \\ -\frac{\mathcal{V}_z}{c^2} & 0 & 0 & -\frac{1}{c^2} \end{pmatrix}$$

⋆ What happens if there are no gravitational nor inertial forces? $\mathcal{U} = 0, \mathcal{V}_i = 0$

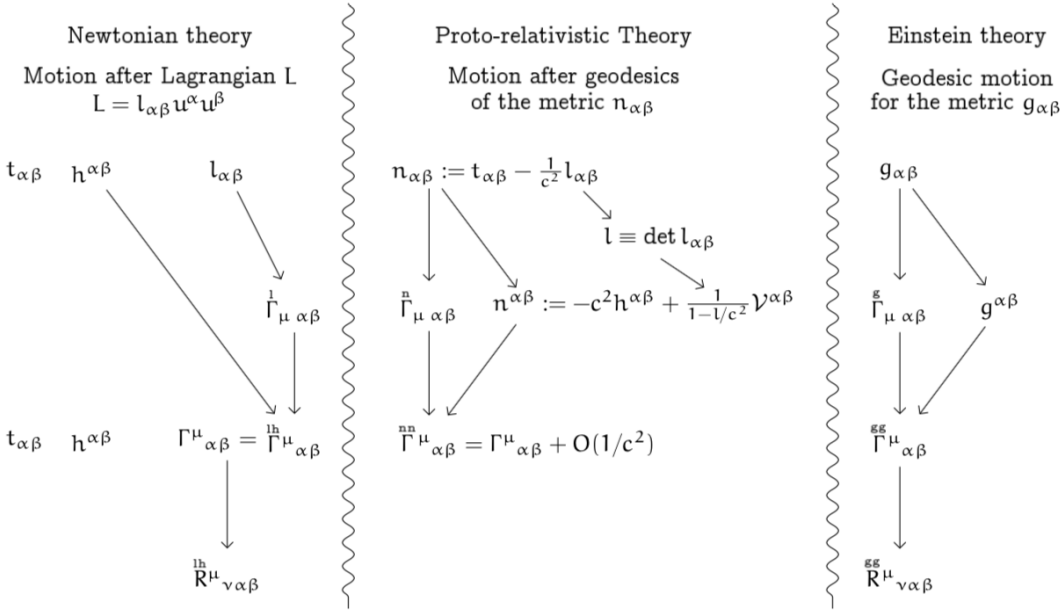
$$n_{\alpha\beta} := t_{\alpha\beta} - \tfrac{1}{c^2} l_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{c^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{c^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{c^2} \end{pmatrix} \equiv \eta_{\alpha\beta}$$

✓ This is the standard metric of Minkowski space, in an inertial RF

✓ Of course, Minkoski space can be also described in noninertial coordinates

⋆ Special relativity from gravitation!

The proto-relativistic theory based in the tensor $n_{\mu\nu}$: a comparative



Is the theory based on the metric $n_{\mu\nu}$ a relativistic theory of gravity?

- No, this is just a 'first order' approximation

$$n_{\alpha\beta} := \begin{pmatrix} 1+\frac{2\mathcal{U}}{c^2} & -\frac{\mathcal{V}_x}{c^2} & -\frac{\mathcal{V}_y}{c^2} & -\frac{\mathcal{V}_z}{c^2} \\ -\frac{\mathcal{V}_x}{c^2} & -\frac{1}{c^2} & 0 & 0 \\ -\frac{\mathcal{V}_y}{c^2} & 0 & -\frac{1}{c^2} & 0 \\ -\frac{\mathcal{V}_z}{c^2} & 0 & 0 & -\frac{1}{c^2} \end{pmatrix} \quad \begin{cases} \mathcal{U}(t,\mathbf{x}) = \varphi(t,x) + \widetilde{\mathcal{U}}(t,\mathbf{x}) \\ \mathcal{V}_i(t,\mathbf{x}) = \qquad\qquad \widetilde{\mathcal{V}}_i(t,\mathbf{x}) \\ \mathcal{U}, \mathcal{V}_i \text{ newtonian potentials} \\ \qquad \text{independent of } c \end{cases}$$

- Beyond that aproximation

$$g_{\alpha\beta} = \begin{pmatrix} 1+\frac{2\mathcal{U}}{c^2} & -\frac{\mathcal{V}_x}{c^2} & -\frac{\mathcal{V}_y}{c^2} & -\frac{\mathcal{V}_z}{c^2} \\ -\frac{\mathcal{V}_x}{c^2} & -\frac{1}{c^2}+\frac{\mathcal{W}_{xx}}{c^4} & \frac{\mathcal{W}_{xy}}{c^4} & \frac{\mathcal{W}_{xz}}{c^4} \\ -\frac{\mathcal{V}_y}{c^2} & \frac{\mathcal{W}_{yx}}{c^4} & -\frac{1}{c^2}+\frac{\mathcal{W}_{yy}}{c^4} & \frac{\mathcal{W}_{yz}}{c^4} \\ -\frac{\mathcal{V}_z}{c^2} & \frac{\mathcal{W}_{zx}}{c^4} & \frac{\mathcal{W}_{zy}}{c^4} & -\frac{1}{c^2}+\frac{\mathcal{W}_{zz}}{c^4} \end{pmatrix} \quad \begin{cases} \mathcal{U}(t,\mathbf{x}) = \varphi(t,x) \\ \mathcal{V}_i(t,\mathbf{x}) = \Phi_i(t,x) \\ \mathcal{W}_{ij}(t,\mathbf{x}) = \Phi_{ij}(t,x) \\ \mathcal{U}, \mathcal{V}_i, \mathcal{W}_{ij} \text{ relativistic} \\ \quad \text{potentials posibly} \\ \quad \text{dependiing on } c \end{cases}$$

⋆ New Expected effects

✓ Gravimagnetism, Spatial curvature, Gravitational waves, ...

Several curious consequences of this play

- ⋆ As gravitation also affects (unavoidably) the time, this Newton-Cartan-like theory of gravitation must be modified, even prior to knowledge of Special Relativity
- ⋆ Alternatively, this 'Newton-Cartan-like theory' might even lead to Special Relativity (with no reference to electromagnetism).

• Once Special Relativity changed the old chronogeometric structure, this Proto-Relativistic 'Newton-Cartan-like' theory would have made a new relativistic theory of gravitation much easier to build.

Schwarzstein, a mythical character

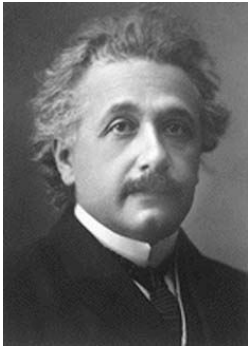
- Can you imagine something similar had been actually done between 1903 and 1917 by some researcher, had Einstein never lived?



Ernst Mach



Karl Schwarzschild



Albert Einstein

Karl A. Schwarzstein

Coda: If the concept of connection had been available . . .

- Perhaps the 'uchronical reconstruction' can/should be considered as an unlikely one.
  - ⋆ But, with a view of how the real history developed, maybe **the** historical route Einstein follow was even more unlikely

Understanding the unfolding of any specific strecht of history greatly benefits from imagining how it could have developed otherwise, if such and such crucial circumstances had been different. *J.M. Lévy-Leblond,2004*

- Should the idea of connection have been known before the time Einstein developed his teory, and would Einstein have learned this through Grossmann, the full development would have been much less difficult and some misunderstanding in the interpretation of the theory —some even living to this day— might have been avoided.
- Essentially, the really fundamental object, both mathematically and physically— is the second level, the connection. But Einstein was limited to work reducing everything to the first level —metric, potentials—.
  - ⋆ But at these level, the newtonian limits of the metric get divorced from the potentials and lose all dynamical content

Coda

. . . Even those who see stars ask "What is a star?", because to see merely with one's eyes is still very little

. . . The "eyes of the mind" must be able to see in the phase space of mechanics, in the space of elementary events of probability theory, in the curved four dimensional space-time of general relativity, in the complex infinite dimensional projective space of quantum theory, . . .

. . . Among other things, modern mathematics is a rigourous training in multidimensional perspective, following a unified program.

Yu. I. Manin, $Mathematics\ and\ Physics$ (Birkhauser, 1981)

Happy scientific '60th and may be more to come, Oscar

Thank you !

Comentaries, observations or criticisms, welcome `mariano.santander@uva.es`

Giga, Presto

IV. The idea of a Classical Space-time

- ⋆ Newton-Cartan gravitation and beyond is presently subject of attention as a model in Solid State Physics:
- ⋆ There was no time to deal with this las part