ALMOST COMPLEX STRUCTURES AND HAMILTONIAN SYMPLECTOMORPHISMS

Moduli Spaces and Geometric Structures
Conference in honor of Oscar Garcia-Prada
on the occasion of his 60th birthday
ICMAT (Madrid), 12 – 16 September 2022

Dietmar SALAMON
ETH Zürich

13 September 2022

GIT: Moment maps

(X, ω, J) closed Kähler manifold, $\mathrm{G} \subset \mathrm{U}(n)$ compact Lie group.

Hamiltonian group action $\mathrm{G} \times X \to X : (u, x) \mapsto ux$

Infinitesimal action $\mathfrak{g} = \mathrm{Lie}(\mathrm{G}) \to \mathrm{Vect}(X) : \xi \mapsto v_\xi$.

$$L_x : \mathfrak{g} \to T_x X, \qquad L_x \xi := v_\xi(x), \qquad x \in X.$$

v_ξ is a Hamiltonian vector field generated by $H_\xi : X \to \mathbb{R}$.

Equivariant moment map $\mu : X \to \mathfrak{g}^*$

$$\omega(L_x \xi, \hat{x}) = \langle d\mu(x)\hat{x}, \xi \rangle, \tag{1}$$

$$\mu(ux) = \mathrm{Ad}^*(u)\mu(x). \tag{2}$$

Marsden–Weinstein quotient

$$X /\!\!/ \mathrm{G} := \mu^{-1}(0)/\mathrm{G}$$

This is a symplectic manifold (in good cases).

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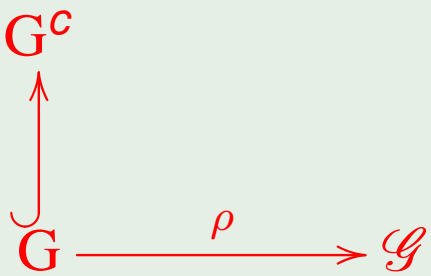


$$\mathrm{Lie}(\mathrm{G}^c) = \mathfrak{g}^c = \mathfrak{g} \oplus \mathbf{i}\mathfrak{g}$$

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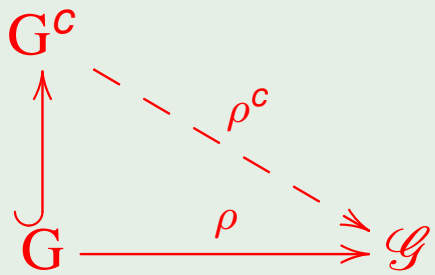
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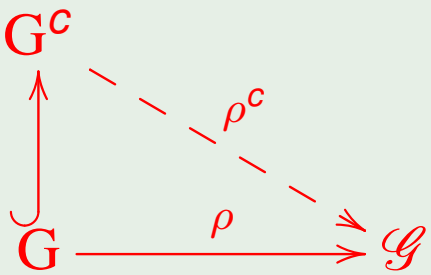
$$\dot{\rho}^c(\xi + \mathbf{i}\eta) = \dot{\rho}(\xi) + \mathbf{i}\dot{\rho}(\eta)$$

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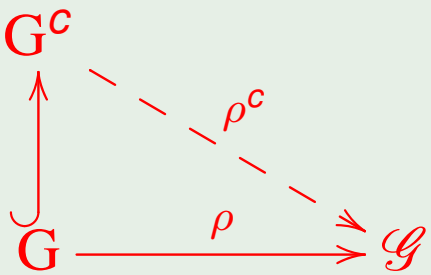
$$L_x^c(\xi + \mathbf{i}\eta) = L_x\xi + JL_x\eta$$

Take $\mathscr{G} = \mathrm{Aut}(X, J)$ $\Longrightarrow$ G^c acts holomorphically on (X, J).

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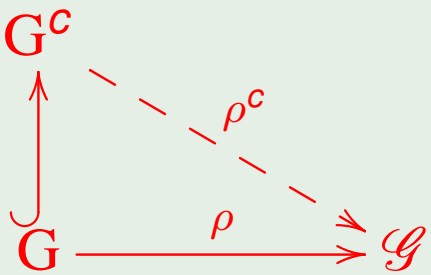
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Complex quotient

$$X/\mathrm{G}^c$$

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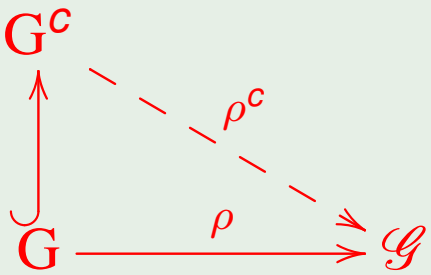
$$X/\mathrm{G}^c$$

In general, this quotient is not Hausdorff.

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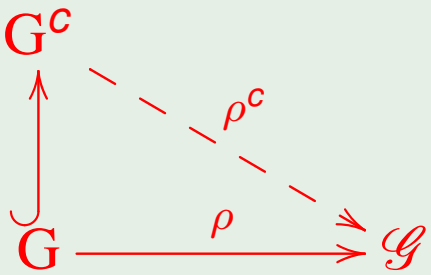
$$X^{\mathrm{ps}}/\mathrm{G}^c$$

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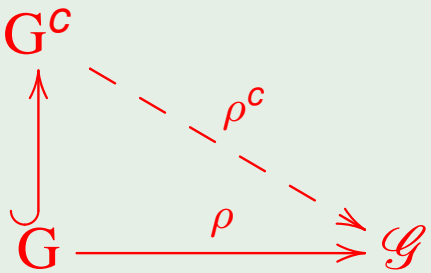
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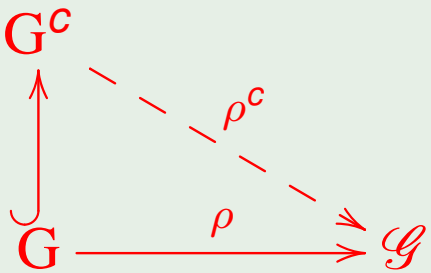
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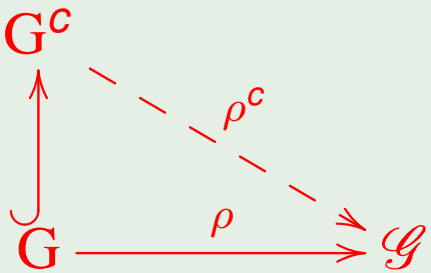
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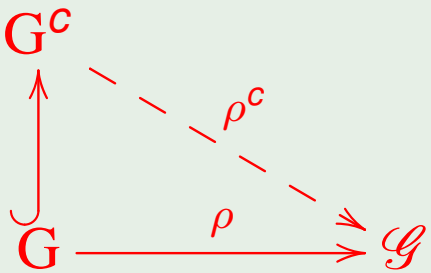
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$x \sim y$ $:\Longleftrightarrow$ $\overline{\mathrm{G}^c(x)} \cap \overline{\mathrm{G}^c(y)} \cap \mu^{-1}(0) \neq \emptyset$.

GIT: Study this quotient (Mumford, Kempf, Ness, Kirwan, ...).
How can one characterize stability?

GIT: The Moment–Weight inequality

Mumford weights

For $x \in X$ and $0 \neq \xi \in \mathfrak{g}$ define

$$w_\mu(x,\xi) := \lim_{t\to\infty} \langle \mu(\exp(\mathbf{i}t\xi)x), \xi \rangle.$$

Choose an invariant inner product on $\mathfrak{g}$.

Theorem (Moment-weight inequality)

For every $x \in X$ we have

$$\sup_{0\neq\xi\in\mathfrak{g}} \frac{-w_\mu(x,\xi)}{|\xi|} \leq \inf_{g\in G^c} |\mu(gx)| \tag{3}$$

and RHS > 0 (i.e. x is μ-**unstable**) implies equality.

Proof: Kempf–Ness, Chen–Sun, Georgoulas–Robbin–S.

Corollary (Hilbert–Mumford criterion)

$$x \text{ is } \mu\text{-unstable} \iff w_\mu(x,\xi) < 0 \text{ for some } 0 \neq \xi \in \mathfrak{g},$$

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$$d\Phi_x(g)\widehat{g} = -\langle \mu(g^{-1}x), \mathrm{Im}(g^{-1}\widehat{g}) \rangle, \qquad \Phi_x(u) = 0$$

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- Φ_x descends to G^c/G (a Hadamard space).
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The square of the moment map $f := \frac{1}{2}|\mu|^2 : X \to \mathbb{R}$

A negative gradient flow lines of f is a solutions $x : \mathbb{R} \to X$ of

$$\dot{x} = -JL_x\mu(x), \qquad x(0) = x_0.$$

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- $x \in X^{ss}$ if and only if $w_\mu(x, \xi) \geq 0$ for all $\xi \in \mathfrak{g}$.

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$$\mu(x_0) = \xi + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \mathrm{ad}_\xi^{k-1}(\mu(x_0)).$$

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Almost complex structures: The moment map

(M,ω) closed symplectic $2n$-manifold, $\rho := \omega^{2n}/n!$ volume form.

$\mathscr{J}(M,\omega) := \{J \in \Omega^0(M, \mathrm{End}(TM)) \mid J^2 = -\mathbb{1}, \text{ comp with } \omega\}$.

$\mathscr{J}(M,\omega)$ is an infinite-dimensional Kähler manifold

$T_J\mathscr{J}(M,\omega) = \{\widehat{J} \in \Omega^0(M, \mathrm{End}(TM)) \mid \widehat{J}J + J\widehat{J} = 0, \widehat{J} = \widehat{J}^*\}$

symplectic form: $\Omega_J(\widehat{J}_1, \widehat{J}_2) := \frac{1}{2}\int_M \mathrm{tr}(\widehat{J}_1 J \widehat{J}_2)\rho$.

complex structure: $\widehat{J} \mapsto J\widehat{J}$ (compatible with $-\Omega$).

Theorem (Quillen, Fujiki, Donaldson 1997)

The action of $\mathrm{Ham}(M,\omega)$ on $\mathscr{J}(M,\omega)$ is Hamiltonian with the moment map $\mathscr{J}(M,\omega) \to \Omega^0_\rho(M) : J \mapsto S_{\omega,J} - c_\omega$.

$S_{\omega,J} := (2\mathrm{Ric}_{\rho,J} \wedge \omega^{n-1}/(n-1)!)/\rho$ is the **scalar curvature**.

Moment map identities (with $\iota(v_H)\omega = dH$, $\mathcal{L}_v J = \nabla_v J + [J, \nabla v]$):

$$\Omega_J(\widehat{J}, \mathcal{L}_{v_H} J) = \int_M \widehat{S}_\omega(J,\widehat{J}) H \rho, \tag{4}$$

$$\Omega_J(\mathcal{L}_{v_F} J, \mathcal{L}_{v_G} J) = \int_M S_{\omega,J}\{F,G\}\rho. \tag{5}$$

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Complex structures: analogy with GIT (Donaldson)

GIT

- compact Kähler mfld (X, ω, J)
- compact Lie gp G, $\mathfrak{g} = \mathrm{Lie}(G)$
- moment map $\mu : X \to \mathrm{Lie}(G)$

csc Kähler geometry

- $(\mathscr{J}(M, \omega), \Omega, \widehat{J} \mapsto J\widehat{J})$
- $\mathrm{Ham}(M, \omega), \Omega^0(M)/\mathbb{R}$

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$$d\Phi_x(g)\hat{g} = -\langle \mu(g^{-1}x), \mathrm{Im}(g^{-1}\hat{g}) \rangle$$

KN flow = gradient flow

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KN flow

csc Kähler geometry

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Almost complex structures: exact isotopy

$\mathrm{Lie}(\mathrm{Ham}(M,\omega)) \cong \Omega^0_\rho(M)$, complexified Lie algebra: $\Omega^0_\rho(M,\mathbb{C})$. There is no complexified Lie group! However, there is an infinitesimal action of the complex Lie algebra $\Omega^0_\rho(M,\mathbb{C})$ on the space $\mathscr{J}(M,\omega)$ via the operators $\mathbb{L}_J : \Omega^0_\rho(M,\mathbb{C}) \to T_J\mathscr{J}(M,\omega)$,

$$\mathbb{L}_J(F+\mathbf{i}G) := \mathcal{L}_{v_F}J + J\mathcal{L}_{v_G}J.$$

Definition (Exact isotopy)

$J, J' \in \mathscr{J}(M,\omega)$ are called **exact isotopic** iff there exists a path $[0,1] \to \mathscr{J}(M,\omega) \times \Omega^0_\rho(M,\mathbb{C}) : t \mapsto (J_t, F_t + \mathbf{i}G_t)$ such that

$$\partial_t J_t = \mathcal{L}_{v_{F_t}} J_t + J_t \mathcal{L}_{v_{G_t}} J_t \tag{6}$$

for $0 \le t \le 1$ and $J_0 = J$, $J_1 = J'$.

Exact isotopy defines an equivalence relation on $\mathscr{J}(M,\omega)$. Denote the equivalence class of $J \in \mathscr{J}(M,\omega)$ by

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(A consequence of) The Uniqueness Conjecture (U)

Given a function $t \mapsto F_t + \mathbf{i}G_t$, equation (6) has at most one solution $t \mapsto J_t$ for each initial condition $J_0 = J$.

The Existence Conjecture (E)

For every pair (J, G) there exists a solution $t \mapsto (J_t, F_t + \mathbf{i}G_t)$ of equation (6) such that $J_0 = J$ and $G_0 = G$.

For each $J \in \mathscr{J}(M, \omega)$ the space

$$\mathfrak{g}_J := \ker(\mathbb{L}_J) = \left\{ F + \mathbf{i}G \in \Omega^0_\rho(M, \mathbb{C}) \,\middle|\, \mathcal{L}_{v_F} J + J\mathcal{L}_{v_G} J = 0 \right\}$$

is a finite-dim'l Lie algebra with the complex Poisson bracket.

The Dimension Conjecture (D)

If $J, J' \in \mathscr{J}(M, \omega)$ are exact isotopic, then $\dim(\mathfrak{g}_J) = \dim(\mathfrak{g}_{J'})$.

Remark: Conjectures (E) and (D) hold in the integrable case. Conjecture (U) holds when $F_t = 0$ and G_t is independent of t.
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For every pair (J, G) there exists a solution $t \mapsto (J_t, F_t + \mathbf{i}G_t)$ of equation (6) such that $J_0 = J$ and $G_0 = G$.

For each $J \in \mathscr{J}(M, \omega)$ the space

$$\mathfrak{g}_J := \ker(\mathbb{L}_J) = \left\{ F + \mathbf{i}G \in \Omega^0_\rho(M, \mathbb{C}) \,\middle|\, \mathcal{L}_{v_F} J + J \mathcal{L}_{v_G} J = 0 \right\}$$

is a finite-dim'l Lie algebra with the complex Poisson bracket.

The Dimension Conjecture (D)

If $J, J' \in \mathscr{J}(M, \omega)$ are exact isotopic, then $\dim(\mathfrak{g}_J) = \dim(\mathfrak{g}_{J'})$.

Remark: Conjectures (E) and (D) hold in the integrable case. Conjecture (U) holds when $F_t = 0$ and G_t is independent of t.

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Theorem 1 (assuming (U) and (D)).

Let $(s,t) \mapsto (J_{s,t}, F_{s,t} + \mathbf{i}G_{s,t})$ and $t \mapsto H_t + \mathbf{i}K_t$ be smooth maps satisfying the equations $\partial_t J_{0,t} = \mathcal{L}_{v_{H_t}} J_{0,t} + J_{0,t}\mathcal{L}_{v_{K_t}} J_{0,t}$ and

$$\partial_s J = \mathcal{L}_{v_F} J + J\mathcal{L}_{v_G} J. \tag{7}$$

Then there exists a unique smooth map $(s,t) \mapsto H_{s,t} + \mathbf{i}K_{s,t}$ such that $H_{0,t} + \mathbf{i}K_{0,t} = H_t + \mathbf{i}K_t$ and

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Corollary

If $J, J' \in \mathscr{J}(M,\omega)$ are exact isotopic, then $\mathfrak{g}_J$ and $\mathfrak{g}_{J'}$ are isomorphic as complex Lie algebras.

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Theorem 1 with $J, F + \mathbf{i}G, H + \mathbf{i}K$ independent of t.

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Let $J_0 \in \mathscr{J}(M,\omega)$. Assume $\mathfrak{g}_{J_0} = 0$ and that $\mathrm{Ham}(M,\omega)$ acts freely on $\mathscr{G}^c(J_0)$. Define the quotient space

$$\mathscr{H} := \mathscr{G}^c(J_0)/\mathrm{Ham}(M,\omega).$$

Its tangent space at $p = [J] \in \mathscr{H}$ is the quotient space

$$T_p\mathscr{H} = \frac{\left\{\mathcal{L}_{v_F}J + J\mathcal{L}_{v_G}J \,\middle|\, F, G \in \Omega^0_\rho(M)\right\}}{\left\{\mathcal{L}_{v_F}J \,\middle|\, F \in \Omega^0_\rho(M)\right\}} \overset{\iota_p}{\cong} \Omega^0_\rho(M).$$

Here $\iota_p : \Omega^0_\rho(M) \to T_p\mathscr{H}$ is the isomorphism

$$\iota_p(G) := [J\mathcal{L}_{v_G}J].$$

A Riemannian metric on $\mathscr{H}$

For $p = [J] \in \mathscr{H}$ and $v = \iota_p(G) \in T_p\mathscr{H}$ (with $\int_M G\rho = 0$) define

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A generalized Mabuchi functional

Theorem 2

$\mathscr{H}$ is an infinite-dimensional symmetric space with nonpositive sectional curvature.

Proof of Theorem 2 (based on Theorem 1):

1: A curve $t \mapsto [J_t] \in \mathscr{H}$ is a geodesic if and only if

$$\partial_t J_t = \mathcal{L}_{v_{F_t}} J_t + J_t \mathcal{L}_{v_{G_t}} J_t, \qquad \partial_t G_t + \{F_t, G_t\} = 0. \tag{10}$$

2: The Riemann curvature tensor on $\mathscr{H}$ is given by

$$R_p(v_1, v_2)v_3 = \iota_p(\{\{G_1, G_2\}, G_3\}) \tag{11}$$

for $p = [J] \in \mathscr{H}$ and $v_i = \iota_p(G_i) \in T_p\mathscr{H}$.

Theorem 3 (The generalized Mabuchi functional)

There exists a unique functional $\mathscr{M} : \widetilde{\mathscr{H}} \to \mathbb{R}$ such that $\mathscr{M}([J_0]) = 0$ and, for $p = [J] \in \mathscr{H}$ and $v = \iota_p(G) \in T_p\mathscr{H}$,

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There exists a unique functional $\mathscr{M} : \widetilde{\mathscr{H}} \to \mathbb{R}$ such that $\mathscr{M}([J_0]) = 0$ and, for $p = [J] \in \mathscr{H}$ and $v = \iota_p(G) \in T_p\mathscr{H}$,

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This functional is convex along geodesics.

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$\mathscr{H}$ is an infinite-dimensional symmetric space with nonpositive sectional curvature.

Proof of Theorem 2 (based on Theorem 1):

1: A curve $t \mapsto [J_t] \in \mathscr{H}$ is a geodesic if and only if

$$\partial_t J_t = \mathcal{L}_{v_{F_t}} J_t + J_t \mathcal{L}_{v_{G_t}} J_t, \qquad \partial_t G_t + \{F_t, G_t\} = 0. \tag{10}$$

2: The Riemann curvature tensor on $\mathscr{H}$ is given by

$$R_p(v_1, v_2)v_3 = \iota_p(\{\{G_1, G_2\}, G_3\}) \tag{11}$$

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Three questions

Question 1 (Moment-weight inequality)

Let $J \in \mathscr{J}(M,\omega)$ and let $H \in \Omega^0(M)$ such that $\int_M H\rho = 0$. Suppose that the equation $\partial_t J_t = J_t \mathcal{L}_{v_H} J_t$, $J_0 = J$, has a solution $t \mapsto J_t \in \mathscr{J}(M,\omega)$ for $0 \le t < \infty$. Does this imply the inequality

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Let $J \in \mathscr{J}(M,\omega)$. Under which (stability) condition does there exist an element $J' \in \mathscr{G}^c(J)$ such that $S_{\omega,J'} = c_\omega$?

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Case 1: Geodesics. If $\partial_t G_t + \{F_t, G_t\} = 0$, then

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This is a version of the Monge–Ampère equation.
Case 2: The Calabi flow. If $F_t = 0$ and $G_t = c_\omega - S_{\omega, J_t}$, then

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$$\partial_t \omega_t' = d(dH_t \circ J_t'), \quad \partial_t J_t' = N_{J_t'}(\cdot, v_t'), \quad \iota(v_t')\omega_t' = dH_t. \tag{14}$$

Case 1: Geodesics. If $\partial_t G_t + \{F_t, G_t\} = 0$, then

$$\partial_t H_t + |dH_t|^2_{\omega_t', J_t'} = 0. \tag{15}$$

This is a version of the Monge–Ampère equation.
Case 2: The Calabi flow. If $F_t = 0$ and $G_t = c_\omega - S_{\omega, J_t}$, then

$$H_t = c_\omega - S_{\omega_t', J_t'}. \tag{16}$$

Note: If J is integrable, then $J_t' = J$ for all t. So these are the standard equations for geodesics and the Calabi flow in the space of Kähler potentials.

The integrable case - Donaldson's construction

Assume $J \in \mathscr{J}_{\mathrm{int}}(M,\omega)$ and $H^1(M,\mathbb{R}) = \{0\}$.
Then $\mathfrak{g}_J \cong \mathrm{Vect}(M,J) := \{v \in \mathrm{Vect}(M) \mid \mathcal{L}_v J = 0\}$. Define

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Lemma 1 (assuming (U))

Let $J' \in \mathscr{J}(M,\omega)$. Then J' is exact isotopic to J if and only if there exists a pair $(\psi, h) \in \mathscr{X}_J$ such that $\psi^* J = J'$.

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THANK YOU!