Some Rationality questions regarding Hyper-elliptic curves:

S. Ramanan

Greetings

It gives me great pleasure to take part in a conference in honour of one of my closest friends.
Often I could spend time with him at various locations.
We collaborated formally as well.
I wish him and Claudia many, many happy and healthy years!

My talk involves

- Pencil of Quadrics
- Hyper-elliptic Curves
- Vector Bundles on them
- Brauer Group

Pencil of quadrics

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A *pencil of quadrics* is a linear imbedding $q : U \to S^2(V)^*$. In other words, for every $u \in U$, a quadratic form q_u on V.

We will assume it is 'generic' in the following sense. For most $u \in U$ the corresponding quadratic form is non-degenerate. Those which are not, will be of rank 1 less and of multiplicity 1. We may simply say that degenerate quadrics of the pencil form a proper smooth subvariety of P.

If $q : U \to S^2(V^*)$ is a pencil, for every $u \in U$ we have a map $V \to V^*$ and taking its deteminant we get an element of $S^{2g+2}(U^*)$. Its multiple by $(-1)^{g+1}$ is called the *discriminant* of the pencil.

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Our genericity assumption is that the discriminant defines a smooth K-subvariety D of P of length $2g + 2$.

There is also a (0-dimensional) subvariety N of $P(V)$ of length $2g + 2$ given by $\{v : q_u(v, V) = 0 \text{ for some } u \in P\}$. Over a suitable L, this consists of 1-dimensional nulspaces N_u of the singular quadrics of the pencil.

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Then with $a \neq b \in P$, the spaces N_a and N_b are orthogonal for both q_a and q_b and hence for all quadrics of the pencil. Thus we get a direct sum decomposition $V = \oplus_{u \in D} N_u$ and the quadrics are all of the form $\sum a_i x_i^2$ in a suitable basis.

The smooth K-sub-variety N of $P(V)$ length $2g+2$ becomes the finite set $\{N_u\}$ over L. This is isomorphic to D. However, the sheaf $\mathcal{O}_V(1)$ on D is, under this isomorphism, actually $\mathcal{O}_U(2)$ on N.

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Thus on identifying D with N, we get a quadratic map of $\Gamma(N, \mathcal{O}_V(1))$ into $\Gamma(D, \mathcal{O}_U(1))$, which, over L, becomes the quadratic map $(\sum x_i) \to (\sum x_i^2)$ with respect to a suitable basis.

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There is a natural linear imbedding of $U^* = \Gamma(P, \mathcal{O}(1))$ in $\Gamma(D, \mathcal{O}(1))$ and composing with the above quadratic map, we get the pencil we started with.

Hyper-elliptic curves

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Given any element f of $S^{2g+2}(U^*)$, we consider the 2-sheeted covering C of P namely, $Spec(\mathcal{O} \oplus \mathcal{O}(-g-1))$ which is a coherent sheaf of Algebras, the multiplication being given by the pairing $\mathcal{O}(-g-1) \times \mathcal{O}(-g-1) \to \mathcal{O}$ defined by f.

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The isomorphism of the sheaf of algebra which is Id on $\mathcal{O}$ and $-Id$ on $\mathcal{O}(-g-1)$ gives an involution ι of C with P as the quotient. The ι-fixed divisor W is called the *Weierstrass divisor*. Over L it consists of $2g+2$ Weierstrass points. The spaces W and D may be identified by the quotient map: $\pi : C \to P$.

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The pull-back h of $\mathcal{O}(1)$ to C is of degree 2 and is defined over K. It is easy to see that for every line bundle ξ of degree d on C, we have $\xi \otimes \iota^*(\xi) \simeq h^d$.

Thus every pencil gives rise to a hyper-elliptic curve. However, while every (smooth) $f \in S^{2g+2}(U^*)$ gives rise to a hyper-elliptic curve over k, it is not true that every such element f is the discriminant of a pencil and hence every hyper-elliptic curve does not arise from a pencil of quadrics.

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If we assume that there exists a line bundle of odd degree over K, then for every integer r there is a line bundle of degree r. It is easily seen that this is equivalent to requiring that over an extension of K of odd degree there exists a rational point in C.

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In particular, we may assume that ξ is a line bundle of degree $2g-1$. Then the restriction of ξ to W has as its square $\xi \otimes \iota^*\xi \simeq h^{2g-1}$. This gives a quadratic map $V = \Gamma(W, \xi) \to \Gamma(W, h^{2g-1})$

We have a natural connecting homomorphism $\Gamma(W, h^{2g-1}) \to H^1(C, h^{g-2})$ given by the exact sequence

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It can be checked (for example, going over to L) that the discriminant of this pencil is f. Thus although every hyper-elliptic curve is not one associated to a pencil in general, it is so if it has a K-rational line bundle of odd degree, in particular if it admits a rational point.

Vector bundles over a hyper-elliptic curve

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We then have a restriction homomorphism $\Gamma(C, L(\alpha))$ to $\Gamma(W, \xi)$. The kernel is $\Gamma(C, L(\alpha) \otimes h^{-(g+1)})$ which is zero since the degree of this line bundle is $(2g-1)-(2g+2)$ which is negative.

We have thus associated to every α, a $2g-1+1-g=g$-dimensional subspace $S(\alpha)$ of V. The pencil restricted to $S(\alpha)$ clearly factors through $\Gamma(C, L(\alpha)\otimes \iota L(\alpha)) = \Gamma(C, h^{2g-1}) \to \Gamma(W, h^{2g-1})$ and hence maps to 0 by the boundary homomorphism of the above exact sequence.

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In other words, we have a map $J \to I_g$, where we denote by I_d the variety of subspaces of dimension d which are isotropic for all quadrics of the pencil.

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Conversely if we take a subspace S of dimension g which is isotropic for the pencil, to each quadric of the pencil we may associate its orthogonal complement (a vector space $S^\perp$ of dimension $g+2$) and a splitting of $S^\perp/S$ as a direct sum of two 1-dimensional subspaces. This gives effectively a line bundle on C.

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Theorem. *The moduli space of stable vector bundles over C of dimension 2 and determinant ξ is canonically isomorphic to I_{g-1}.*

If E is a vector bundle of dimension 2 with determinant ξ, we associate the (-1)-eigenspace for ι in $\Gamma(E \otimes \iota^* E)$. By Atiyah-Bott fixed point theorem, its dimension is $g-1$. Restricting it to W, we get a map $M_2(\xi) \to I_{g-1}$. As in the case of line bundles, we obtain the theorem stated above.

If the genus of the curve is 2, (which is always hyper-elliptic) we will re-state the above. Assuming it admits a line bundle of odd degree, there is a pencil of quadrics in P^5, to which it is associated. The moduli of line bundles of any fixed degree, is isomorphic to the locus of lines contained in all the quadrics of the pencil.

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In any case, it was proved over the algebraic closure long ago by Narasimhan and myself that corresponding to every vector bundle E as above, there exists a divisor in the Jacobian of degree 1, every point α of which admits a non-zero section of $E \otimes \alpha$ and that this association gives an isomorphism of M_2 with $P(\Gamma, 2\theta)$.

There is also a correspondence between $M' = M_2(\mathcal{O}(-p))$ and $M = M_2$, where p is a rational point on C. This is given by the closed sub-variety of their product given by $\{(F, E) : Hom(F, E) \neq 0\}$. The projection of this variety on M' has as fibre over $F \in M'$ the projective line $P(F_p)$, namely the kernel of the induced map $F_p \to E_p$.

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On the other hand, the fibre over $E \in M_2$ is the conic $P(E_p^*)$ if E is stable, which degenerates to a pair of lines if E is only semi-stable but not stable.

The non-stable locus in M_2 is the Kummer (quartic) surface imbedded in P^3.

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Since the Grassmannian is in this case a quadric in P^5, we get the above theorem in terms of this correspondence.

BRAUER GROUP

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The tensor product of two csa's is again one such. This defines an operation on the set of csa's. The tensor product of a csa and its opposite is a matrix algebra.

Going modulo matrix algebras, one defines a group called the *Brauer group* associated to K.

It is a torsion group. If A is a csa, the (unique) division algebra over which it is a matrix algebra has dimension a square. Its square root is called the *index* of A. The order of A in the Brauer group is called its *period*.

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One also defines the Brauer group over a smooth projective curve – in fact, even over a scheme, as follows. Take any vector bundle with an algebra structure which over the residue field of any point is a csa and define multiplication in the obvious way.

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If the variety admits a section (which it does if it is actually an endomorphism algebra of a vector bundle) then its vector bundle dual has a divisor at each point and the direct image on C defines a vector bundle with the Severi-Brauer variety as its projectivisation. In other words, Severi-Brauer varieties define elements of the Brauer group and it is trivial if and only if it admits a section.

TWISTING OF BUNDLES

Jaya Iyer and Parimala Raman proved the following theorem.

Theorem. *Under the assumptions: C is a hyper-elliptic curve of genus 2 admitting a rational point and K is a totally imaginary number field, any element of the Brauer group of period 2 has index 2.*
They used the concept of *twisted vector bundles* introduced by Lieblich in order to prove this theorem.

Technically, one considers gerbes over C in order to define twisting. One simple way of defining them is the following. With respect to a covering U_i (in the etale topos, for example) we may define them as follows. Take an element of β in $H^2(C, \mu(n))$. We have the isomorphism $H^2(C, \mu_n) = H^2(k, \mu_n) \oplus H^1(k,_n Pic) \oplus Z/n$, Hence β defines an element of $H^1(k,_n Pic)$.

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Since ${}_nPic$ acts on M_2 , this defines another variety by Galois twist. This may be denoted M_2^{β}.

We may represent β by $c_{ijk} \in \Gamma(U_{ijk})$. Now consider $m_{ij} \in \Gamma(U_{ij}, GL(n))$ such that $m_{ij}m_{jk}m_{ki} = c_{ijk}$ for all i, j, k.

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Upto an isomorphism this depends only on β and may be called a β-twisted (or simply a β-)vector bundle. These may be considred as points of the twisted varerty. Of course if c_{ijk} is a coboundary, then this represents a vector bundle over C of rank n.

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Moreover If $\beta, \beta' \in H^2(C, \mu(n))$ then there is a natural tensor product of a β bundle and a β'-bundle. In particular, if $\beta.\beta' = 1$, then the tensor product is actually a vector bundle of dimension n^2 and comes with an algebra structure – hence an element of the Brauer group of index that divides n.

If $a_{ij} \in \Gamma(\mathcal{O}^*)$ satisfying the same relation to β as above, then a_{ij}^n defines a line bundle L which is unique upto an n-th power of a line bundle. We denote the gerbe defined by a_{ij} as $L^{1/n}$.

The above theorem may therefore be proved by showing that under the conditions stated in the theorem M_2^β has a K-rational point.

They do this using the correspondence between M_2 and M_2' mentioned above, and proving a similar correpondence in the case of twisied varieties.

From what we remarked above, if E is a β-twisted bundle, then its endomorphisms constitute an Azumaya algebra. We know that any element of $\Gamma(C, \mu_n)$ may be considered as an element of the Brauer group of period (order) dividing n. This is given by β.

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It is then clear that if there exists a β-twisted bundle defined over K then the element of the Brauer group defined by β is represented by an algebra of rank n^2 so that its index is divisible by n.

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Thus the period index problem is related closely to the existence of a K-rational point of the twisted moduli space. We will hereafter assume that $n = 2$ and $g = 2$ and that K is a totally imaginary number field and show that if the period is 2 then so is the index.

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Note that the moduli space $M_2(\mathcal{O})$ is P^3 over the algebraic closure. Hence it is a Severi-Brauer variety. But it has a rational point, namely the trivial bundle for example, and hence it is itself P^3 over K.

Let us assume that C has a rational point p. Then the moduli space $M_2(\mathcal{O}(p))$ is the intersection of 2 quadrics as we have seen above.

Now we wish to study similar questions over the twisted moduli spaces. Firstly, by a theorem of Colliot-Thelene, any conic bundle over the complement of a quartic surface in P^3 satisfies the Hasse principle, namely if in every associated local field, it has a rational point, it has one over K as well.

Firstly, the β-twisted moduli space $M = M_2^{\beta}(2, \mathcal{O})$ is also isomorphic to P^3. Under our assumption, the localisations of M are isomorphic to untwisted moduli space (perhaps with non-trivial determinant). In any case, they are isomorphic to P^3 and hence admit a rational point. By Hasse principle, so does M and hence it is also P^3.

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Let β be an element of the Brauer group of order 2. They show that there is a rational point in the β-twisted moduli space M_2'. This shows that there exists a β-vector bundle of rank 2 of which the endomorphism bundle defines the element of the Brauer group. Hence its index is 2.

Thus the problem is reduced to showing that the β-twisted moduli space has a rational point. Firstly, the twisted variety M_2 is also proved to be P^3. In particular, it has lots of rational points.

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Finally any rational point in the correspondence variety gives rise to a rational point in M_2'. This gives rise to a rational twisted vector bundle of rank 2 whose endomorphism variety (of dimension 4) gives the element β. This proves the period-index theorem claimed above.

THANK YOU FOR YOUR

ATTENTION!