Geometry and topology of wild character varieties

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Moduli spaces and geometric structures
Conference in honour of Oscar Garcia-Prada
on the occasion of his 60th birthday
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Outline

- Based on joint works with with **Diaconescu and Donagi**, and with **Chuang, Diaconescu, Donagi, and Nawata**.
- Review of wild character varieties.
- Topology - weighted Poincaré polynomials.
- Calabi-Yau enumerative geometry.
- Refined Chern-Simons theory of torus knots and calculations.

Wild character varieties

Character varieties (i)

Character varieties = special moduli spaces of matrices.

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- tame,
- wild.

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- tame, $\leftarrow$ logarithmic poles
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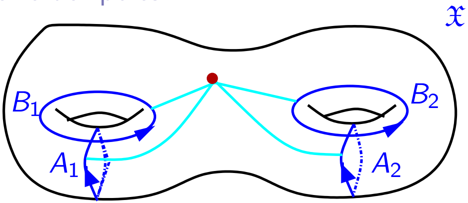

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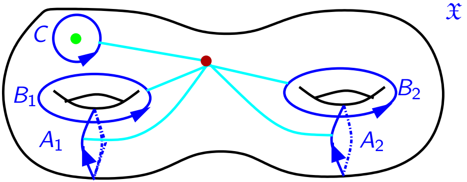

$$\prod_{i=1}^{g} [A_i, B_i] \cdot C_1 \cdots C_k = 1$$

Character varieties (ii)

Depend on:

Background data:

- $\mathfrak{X}$ - a smoooth projective curve/$\mathbb{C}$.
- $D = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k\}$ - distinct points.
- G - a complex reductive group, $\mathfrak{g} = \mathrm{Lie}(G)$.

Invariants:

- $\mathrm{Q} = \{Q_1, \ldots, Q_k\}$ - irregular type at D.
- Fixed conjugacy classes of formal monodromy.

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Study monodromies of (V, ∇) with V a principal G-bundle and ∇ meromorphic flat connection with poles at most at D.

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- $\mathrm{Q} = \{Q_1, \ldots, Q_k\}$ - irregular type at D.
- Fixed conjugacy classes of formal monodromy.

Study monodromies of (V, ∇) when ∇ has polar parts Q_i at x_i and formal monodromy in the given conjugacy classes.

Irregular type (i)

Definition

A G-valued irregular type *is a Laurent polynomial*

$$Q = \frac{A_{n-1}}{z^{n-1}} + \cdots + \frac{A_1}{z}, \ A_i \in \mathfrak{t},$$

where $\mathfrak{t} \subset \mathfrak{g}$ is a Cartan subalgebra.

(V, ∇) has irregular type Q at $x_i \in D$ *if locally near x_i*

$$\nabla = d + dQ \ mod \ z^{-1}\mathfrak{g}\{z\}dz$$

in an analytic local trivialization of V near x_i and for some finite root z of a local coordinate on $\mathfrak{X}$ centered at x_i

Note: Up to a Puiseux gauge transformation the ≤ -2 polar part of ∇ always has coefficients in $\mathfrak{t} \subset \mathfrak{g}$.

Irregular type (ii)

Consider $\widehat{\mathfrak{X}}$ - the **real oriented blow up** of $\mathfrak{X}$ at D

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Consider $\widehat{\mathfrak{X}}$ - the **real oriented blow up** of $\mathfrak{X}$ at D

- each x_i is replaced by a circle ∂_i parametrizing real oriented directions emanating from x_i;
- $\widehat{\mathfrak{X}}$ is a surface with boundary $\partial \widehat{X} = \partial_1 \sqcup \ldots \sqcup \partial_k$.

Irregular type (ii)

Consider $\widehat{\mathfrak{X}}$ - the **real oriented blow up** of $\mathfrak{X}$ at D

Boalch: An irregular type Q on (X, D) decorates $\widehat{\mathfrak{X}}$ with:

- A connected reductive group $L_i \subset G$, the centralizer of Q_i in G.
- A finite set $\mathrm{A}_i \subset \partial_i$ of singular/anti-Stokes directions for all $i = 1, \ldots, k$.
- For each $d \in \mathrm{A}_i$ a unipotent group $\mathrm{Sto}_d \subset G$, normalized by L_i.

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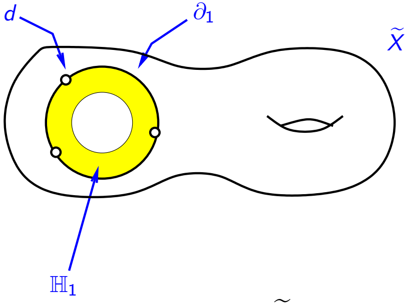
Note: These give rise to a new punctured surface surface $\widetilde{X}$ defined by puncturing $X = \mathfrak{X} - D$ once near each singular direction..

Irregular type (iii)

Concretely:

- Glue an annulus $\mathbb{H}_i = (0,1] \times \partial_i$ to each boundary circle $\partial_i \subset \widehat{\mathfrak{X}}$ to obtain a new surface X^+.
- Puncture X^+ at each singular direction $d \in \mathsf{A}_i \subset \partial_i$ to obtain $\widetilde{X}$:



The surface $\widetilde{X}$

Stokes local systems

The analytic covariantly constant local sections in a flat meromorphic G-bundle of irregular type Q on $(\mathfrak{X}, D)$ form a locally constant sheaf on $\widetilde{X}$ of a special type:

Definition

A Stokes G-local system *of irregular type* Q *on* $(\mathfrak{X}, D)$ *is a locally constant principal* G *bundle on* $\widetilde{X}$ *with a locally constant reduction of the structure group to* L_i *on a finite etale cover of* $\mathbb{H}_i$*, so that the local monodromy around each* $d \in \mathrm{A}_i \subset \partial_i$ *is in* Sto_d *for any choice of a base point in* $\mathbb{H}_i$*.*

Riemann-Hilbert correspondence

The **irregular Riemann-Hilbert correspondence** **(Stokes, Malgrange-Sibuya, Deligne-Malgrange, Boalch)** can now be formulated as the following

Theorem

The category of flat meromorphic G-bundles of irregular type Q on $(\mathfrak{X}, D)$ is equivalent to the category of Stokes G-local systems of irregular type Q on $(\mathfrak{X}, D)$.

This leads to our main object of interest: the wild character varieties.

Wild character varieties

Fix $(\mathfrak{X}, D)$, a group G, an irrequalr type Q, and a collection $\mathrm{C} = \{\mathrm{C}_1, \ldots, \mathrm{C}_k\}$, where $\mathrm{C}_i \subset L_i$ is a conjugacy class in L_i.

Definition

The wild character variety of irregular type Q *on* $(\mathfrak{X}, D)$ *is the coarse moduli space* $M_G(\mathrm{Q})$ *of Stokes* G*-local systems of irregular type* Q *on* $(\mathfrak{X}, D)$.
The C*-restricted wild character variety is the locally closed subvariety* $M_G(\mathrm{Q}, \mathrm{C}) \subset M_G(\mathrm{Q})$ *consisting of Stokes local systems whose formal monodromy around* ∂_i *(for a base point in* $\mathbb{H}_i$*) is in* $\mathrm{C}_i \subset L_i$.

Topology

Prelude: counting G-covers (i)

G - a finite group, X a smoooth compact surface of genus g.

Problem: [Riemann,Hurwitz, ...] Count all unramified G-Galois covers of X up to isomorphism.

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Reformulation: Find

$$\begin{pmatrix} \text{orbifold Euler characteristic} \\ \text{of the moduli stack } M_G(X) \\ \text{of flat } G \text{ bundles on } X \end{pmatrix} = \begin{pmatrix} \text{orbifold partition function of} \\ \text{a } 2d \text{ Dijkgraaf-Witten the-} \\ \text{ory with gauge group } G \end{pmatrix}.$$

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Problem: **[Riemann,Hurwitz, ...]** Count all unramified G-Galois covers of X up to isomorphism.

Reformulation: $M_G(X) \cong [\mathrm{Hom}(\pi_1(X), G)/G]$, so find

$$\chi^{\mathrm{orb}}(M_G(X)) = \sum_{[\rho]\in \mathrm{Hom}(\pi_1(X),G)/G} \frac{1}{|\mathrm{Aut}(\rho)|} = \frac{|\mathrm{Hom}(\pi_1(X), G)|}{|G|}.$$

Prelude: counting G-covers (ii)

Formulas:

Frobenius'1896 If $g = 1$, then

$$\chi^{\mathrm{orb}}(M_G(X)) = \frac{|\operatorname{Hom}(\pi_1(X), G)|}{|G|} = |\operatorname{Irrep}(G)|$$

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$$\chi^{\mathrm{orb}}(M_G(X)) = \frac{|\operatorname{Hom}(\pi_1(X), G)|}{|G|} = \sum_{\mu \in \operatorname{Irrep}(G)} \left(\frac{\dim \mu}{|G|}\right)^{2-2g}.$$

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Goal: FInd analogous formulas for $\chi(M_G(\mathrm{Q}, \mathrm{C}))$.

There are two stories . . .

First story: **topology of character varieties**

Fix a genus g curve C and a positive integer N. Consider the character variety

$$\mathcal{M}_{\mathrm{B}}^{N,d} := \left\{ A_1, B_1, \ldots, A_g, B_g \in \mathsf{GL}_N \;\middle|\; \prod_{i=1}^{g} A_i^{-1} B_i^{-1} A_i B_i = e^{\frac{2\pi\sqrt{-1}d}{N}}\,\mathsf{Id} \right\} \Big/\!\!\Big/ \mathsf{GL}_N$$

of (projective) representations of $\pi_1(C)$ into $\mathsf{GL}_N(\mathbb{C})$

Note:

- $\mathcal{M}_{\mathrm{B}}^{N,d}$ is an affine scheme, nonempty, of dimension, $d_N = N^2(2g-2)$, and smooth when $\gcd(d, N) = 1$;
- $H_c^k(\mathcal{M}_B^{N,d}; \mathbb{Q})$ carries a mixed Hodge structure with a weight filtration $W_0 \subset \cdots \subset W_i \subset \cdots \subset W_{2k} = H_c^k(\mathcal{M}_B^{N,d}; \mathbb{Q})$.

First story:

Weighted Poincaré polynomial:

$$\boxed{WP\left(\mathcal{M}_{\mathrm{B}}^{N,d}; u, v\right) = \sum_{i,j} u^{i/2} v^j \dim Gr_i^W H_c^j\left(\mathcal{M}_{\mathrm{B}}^{N,d}\right)}$$

Note:

- $WP(\mathcal{M}_{\mathrm{B}}^{N,d}; u, v)$ is a polynomial in u and v, i.e. $W_{2i}H_c^j = W_{2i+1}H_c^j$.
- using Poincaré duality $WP(\mathcal{M}_{\mathrm{B}}^{N,d}; u, v)$ can be expressed in terms of $h^{p,q}$ for the MHS on $H^\bullet(\mathcal{M}_{\mathrm{B}}^{N,d}; \mathbb{Q})$ instead.

First story:

Conjecture (Hausel,Rodriguez-Villegas)

$$\sum_{\lambda} \prod_{\square\in\lambda} \frac{\left(z^{2l(\square)+1} - w^{2a(\square)+1}\right)^{2g}}{\left(z^{2l(\square)+2} - w^{2a(\square)}\right)\left(z^{2l(\square)} - w^{2a(\square)+2}\right)} x^{|\lambda|}$$

$$= \exp\left(\sum_{n,k} \frac{WP\left(\mathcal{M}_B^{N,d}; w^{2k}, -(zw)^{-2k}\right)}{\left(z^{2k}-1\right)\left(1-w^{2k}\right)(zw)^{-d_N}} \frac{x^{Nk}}{k}\right)$$

Partial extensions:

- C with tame ramifications [Hausel,Letellier,Rodriguez-Villegas]
- C with wild ramifications [Hausel,Mereb,Wong]

A different tack:

Consider the moduli space of Higgs bundles

$$\mathcal{M}_{\mathrm{Dol}}^{N,d} := \left\{ \begin{array}{c} \text{semistable rank } N \text{ degree } d \text{ Higgs bundles } (E, \phi) \\ \phi \in H^0(C, End(E) \otimes K) \text{ Higgs field} \end{array} \right\}$$

Non-Abelian Hodge correspondence: Hyper-Kähler rotation gives a homeomorphism $\mathcal{M}_{\mathrm{B}}^{N,d} \cong \mathcal{M}_{\mathrm{Dol}}^{N,d}$.

Filtrations:

- **weight filtration** $W_0 \subset \cdots \subset W_i \subset \cdots \subset W_{2k} = H_c^k(\mathcal{M}_B^{N,d}; \mathbb{Q})$
- **perverse Leray filtration** $P_0 \subset \cdots \subset P_i \subset \cdots \subset P_k = H_c^k(\mathcal{M}_{Dol}^{N,d}; \mathbb{Q})$ for the Hitchin map $(E, \phi) \mapsto (\mathrm{tr}(\phi), \mathrm{tr}(\phi^2), \ldots, \mathrm{tr}(\phi^N))$.

These match according to the $P = W$ conjecture of de Cataldo, Hausel, and Migliorini.

A different tack:

Conjecture (de Cataldo, Hausel, Migliorini)

$$Gr_{2i}^{W} H_c^j \left(\mathcal{M}_B^{N,d} \right) = Gr_i^P H_c^j \left(\mathcal{M}_{Dol}^{N,d} \right)$$

Using the $P = W$ conjecture can rewrite the weighted Poincaré polynomial of $\mathcal{M}_B^{N,d}$ as the perverse Poincaré polynomial

$$\boxed{PP \left(\mathcal{M}_{\mathrm{Dol}}^{N,d}; u, v \right) = \sum_{i,j} u^i v^j \dim Gr_i^P H_c^j \left(\mathcal{M}_{\mathrm{Dol}}^{N,d} \right).}$$

Second story: **polynomial knot invariants**

Suppose $\Sigma \subset \mathbb{A}^2$ is a reduced rational plane curve with a single singular point ν and let $\pi : \mathbb{A}^2 \to \mathbb{A}^1$ be the projection onto one of the coordinate axes.

Then $\Sigma \simeq$ affine part of a spectral curve for a meromorphic Hitchin system on $\mathbb{P}^1$ with a pole at ∞.

The wild non-abelian Hodge correspondence identifies the (symplectic leaf containing) the compactified Jacobian of the projective completion of Σ with a wild character variety $\mathcal{S}_\Sigma$ on $\mathbb{P}^1$ parametrizing Stokes data at ∞.

Second story

- $L \subset S^3$ - the link of the singular point $\nu \in \Sigma$
- $P_L(a, q) \in \mathbb{Z}\left(q^{1/2}\right)[a^{\pm}]$ - the HOMFLY-PT polynomial of L
- $P_L^{(0)}(q)$ - the a^0 term in $\left(aq^{-1/2}\right)^{m(\nu)} P_L(a, q)$, where $m(\nu)$ is the Milnor number of $\nu \in \Sigma$

Conjecture (Shende, Treumann, Zaslow)

$$P_L^{(0)}(q) = (1 - q)^{b(\nu)} WP\left(\mathcal{S}_\Sigma; q, -1\right)$$

where $b(\nu) =$ number of branches of Σ at ν.

Refined and colored extensions have been studied in [Oblomkov,Rasmussen,Shende] and [Diaconescu,Hua,Soibelman]

Strategy

Note: Knot invariants are very handy because their computation boils down to simple combinatorics.

Goal: Relate Betti numbers of twisted wild character varieties and torus knot invariants.

Strategy: Use the spectral correspondence, Gopakumar-Vafa expansions, and refined Chern-Simons theory.

Steps in the strategy

Twisted wild character variety $\mathcal{S}_{n,\ell,r}$	Weighted Poincaré polynomials
NAH, and $P = W$	
Moduli space $\mathcal{H}_{n,\ell,r}$ of twisted irregular Higgs bundles	Perverse Poincaré polynomials
Spectral correspondence	
Moduli space of pure sheaves F of dimension one on a symplectic surface S	
Gopakumar-Vafa expansion	
Refined stable pairs $\mathcal{O}_Y \xrightarrow{s} F$ of CY3	Refined PT invariants
Large N duality	
Refined Chern-Simons theory of torus knot	Refined Chern-Simons invariants

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Moduli space of twisted irregular flat connections

Main object of study: Moduli space of flat connections with twisted irregular singularity at p on a curve C.

Data:

- (C, p) smooth projective curve with a marked point.
- For any triple (n, ℓ, r) consider the moduli space $\mathcal{C}_{n,\ell,r}$ of pairs (V, A)
  - V is a rank $N = r\ell$, degree 0 vector bundle on C
  - A is a meromorphic connection on V subject to the following conditions:

Moduli space of twisted irregular flat connections

Conditions on (V, A):

- Any proper nontrivial sub-bundle $V' \subset V$ preserved by A has degree $\deg(V') < 0$. (stability condition)
- A has a simple pole at ∞ with residue $\mathbf{1}_N / 2\sqrt{-1}\pi N$.
  (deg=1 in Higgs side)
- A has a pole of order n at the marked point p and there exists a local trivialization of V at p which identifies it to $A_{n,\ell,r}$ up to holomorphic terms where

$$A_{n,\ell,r} = -2dQ_{n,\ell,r} - \ell^{-1} R \frac{dz}{z}$$

is a $\mathfrak{gl}(N, \mathbb{C})$-valued one-form with

$$Q_{n,\ell,r}(z) = \frac{z^{1-n}}{1-n} J_{\ell,r} + \frac{z^{2-n}}{2-n} E_{\ell,r}$$

Moduli space of twisted irregular flat connections

the word, "twisted", refers to the fact that the Laurent tail at p does not have coefficients in a Cartan subalgebra of $\mathfrak{gl}(N,\mathbb{C})$

$$J_{\ell,r} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \mathbf{1}_r & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{1}_r & 0 \end{pmatrix}, \qquad E_{\ell,r} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \mathbf{1}_r \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

The residue R is the diagonal block-matrix

$$R = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & \mathbf{1}_r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & (\ell-1)\mathbf{1}_r \end{pmatrix}.$$

Twisted wild character varieties

The Betti version of $\mathcal{C}_{n,\ell,r}$ was constructed by [Boalch,Yamakawa] as a variety parametrizing twisted Stokes data

- let $\rho : \widetilde{\Delta} \to \Delta$ be the $\ell : 1$ cover $z = w^{\ell}$;
- $\rho^* dQ_{n,\ell,r}$ can be diagonalized and the Stokes transformations can be studied via this covering map;
- there are two types of anti-Stokes directions for twisted wild ramification:
  - complete half-periods: the Stokes groups U_0, U_1, $\dots U_{2n-4}$ are unipotent radicals of parabolic subgroup with Levi H - the group of invertible block diagonal matrices

$$\begin{pmatrix} * & 0 & \cdots & 0 & 0 \\ 0 & * & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & * \end{pmatrix}$$

    with arbitrary $r \times r$ blocks on the diagonal.
  - incomplete half-period: the Stokes group U_{2n-3} is a smaller unipotent subgroup.

Twisted wild character varieties

Let $H(\partial), H(\overline{\partial}) \subset \mathrm{GL}_N$ be the subsets of matrices of block form

$$h = \begin{pmatrix} 0 & * & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & * \\ * & 0 & \cdots & 0 & 0 \end{pmatrix}, \qquad \overline{h} = \begin{pmatrix} 0 & 0 & \cdots & 0 & * \\ * & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & * & 0 \end{pmatrix},$$

respectively.

Note:

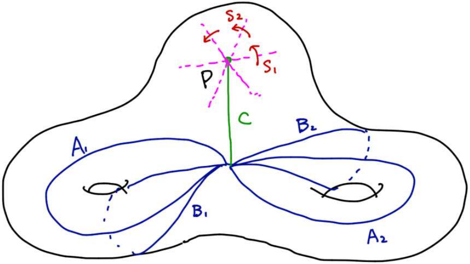
- H acts by conjugation on $H(\partial)$ and $H(\overline{\partial})$;
- The map $h \mapsto h^{-1}$ is an H-equivariant isomorphism $H(\partial) \widetilde{\to} H(\overline{\partial})$.

Twisted wild character varieties

For $\alpha = (\mathsf{A}_i, \mathsf{B}_i, \mathsf{C}, h, S_j) \in \mathcal{A} = \mathsf{GL}_N^{\times 2g+1} \times H(\partial) \times \prod_{j=0}^{2n-3} U_j$ we define a moment map $\mu : \mathcal{A} \to \mathsf{GL}_N \times H(\overline{\partial})$ by setting

$$\mu(\alpha) = \left([\mathsf{A}_1, \mathsf{B}_1] \cdots [\mathsf{A}_g, \mathsf{B}_g] \mathsf{C}^{-1} h \prod_{j=0}^{2n-3} S_j \, \mathsf{C},\ h^{-1} \right)$$

with $[\mathsf{A}_i, \mathsf{B}_i] = \mathsf{A}_i \mathsf{B}_i \mathsf{A}_i^{-1} \mathsf{B}_i^{-1}$.



Twisted wild character varieties

Then the variety of Stokes data for the flat connections parameterized by $\mathcal{C}_{n,\ell,r}$ is given by the quasi-Hamiltonian reduction

$$\mathcal{S}_{n,\ell,r} = \mu^{-1}\left(e^{-2\pi\sqrt{-1}/N}\mathbf{1}_N, \mathcal{C}_{\mathsf{M}^{-1}}\right) / \, \mathsf{GL}_N \times H$$

where M is the formal monodromy

$$\mathsf{M} = \begin{pmatrix} 0 & \mathbf{1}_r & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \mathbf{1}_r \\ \mathbf{1}_r & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Note: $\dim \mathcal{S}_{n,\ell,r} = 2 + r^2\left(n\ell(\ell-1) + 2(g-1)\ell^2\right)$.

Twisted wild character varieties

- We are interested in **weighted Poincaré polynomial**

$$WP_{n,\ell,r}(u,v) = \sum_{i,j} u^{i/2} v^j \dim Gr_i^W H^j(\mathcal{S}_{n,\ell,r})$$

  where $W_k H^j$ is the weight filtration for the mixed Hodge structure on the **cohomology** of $\mathcal{S}_{n,\ell,r}$.
- The weight filtration on cohomology is expected to satisfy the condition $W_{2k} H^j(\mathcal{S}_{n,\ell,r}) = W_{2k+1} H^j(\mathcal{S}_{n,\ell,r})$ for all k, j

Steps in the strategy

Twisted wild character variety $\mathcal{S}_{n,\ell,r}$ — Weighted Poincaré polynomials

NAH, and $P = W$

Moduli space $\mathcal{H}_{n,\ell,r}$ of twisted irregular Higgs bundles — Perverse Poincaré polynomials

Spectral correspondence

Moduli space of pure sheaves F of dimension one on a symplectic surface S

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Twisted irregular Higgs bundles

- (C, p) a curve with marked point
- $D = np \; n \geq 1$
- $M = K_C(D)$

a twisted irregular Higgs bundle (E, ϕ) of rank $N = r\ell$, $\phi : E \to E \otimes K_C(D)$, $D = np$ is subject to the condition that the restriction $\phi|_{np}$ is equivalent to $dQ_{n,\ell,r}(z)$

$$Q_{n,\ell,r}(z) = \frac{z^{1-n}}{1-n} J_{\ell,r} + \frac{z^{2-n}}{2-n} E_{\ell,r}$$

with

$$J_{\ell,r} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \mathbf{1}_r & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{1}_r & 0 \end{pmatrix}, \qquad E_{\ell,r} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \mathbf{1}_r \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Twisted irregular Higgs bundles

- moduli space $\mathcal{H}_{n,\ell,r}$ of semistable meromorphic Higgs bundles of rank $N = r\ell$, degree 1
- (twisted) wild non-abelian Hodge correspondence [Biquard-Boalch, Sabbah, Mochizuki] [Witten]

$$\mathcal{S}_{n,\ell,r} \cong \mathcal{H}_{n,\ell,r}$$

- $\mathcal{H}_{n,\ell,r}$ admits perverse Leray filtration $P_k H^j$ arising from the Hitchin map

Conjecture (Twisted wild version of $P = W$)

$$Gr^W_{2k} H^j(\mathcal{S}_{n,\ell,r}) = Gr^P_k H^j(\mathcal{H}_{n,\ell,r})$$

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Refined Chern-Simons theory of torus knot — Refined Chern-Simons invariants

Spectral (cover) correspondence

Classical version [Hitchin] [Beauville,Narasimhan,Ramanan]
The characteristic polynomial $\det(y\mathbf{1} - \phi) = 0$ of a Higgs field $\phi : E \to E \otimes K_C$ defines an N-sheeted spectral cover

$$\begin{tikzcd} \Sigma \arrow[r, hook] \arrow[dr, "\pi"'] & T^\vee C \arrow[d] \\ & C \end{tikzcd}$$

which is embedded in the symplectic surface $T^\vee C$.

If Σ is an irreducible spectral curve, there is a bijection between isomorphism classes of

- Pairs (E, ϕ) with spectral curve Σ
- Rank one torsion free sheaves L on Σ

Spectral correspondence

Goal: construct a holomorphic symplectic surface S in such a way that

$$\mathcal{H}_{n,\ell,r} \cong \begin{pmatrix} \text{moduli space of stable pure dimension} \\ \text{one sheaves on a symplectic surface } S \end{pmatrix}$$

Construction: [Kontsevich,Soibelman] [Diaconescu,Donagi,P]

- (C, p) a curve with marked point
- $D = np$
- $M = K_C(D)$
- $M = M_0 \leftarrow M_1 \leftarrow \cdots \leftarrow M_{n\ell}$ is a sequence of $n\ell$ successive blow-ups of M with exceptional divisors

$$\Xi_1, \Xi_2, \ldots, \Xi_\ell, \Xi_{\ell+1} \ldots, \Xi_{n\ell} \subset M_{n\ell}.$$

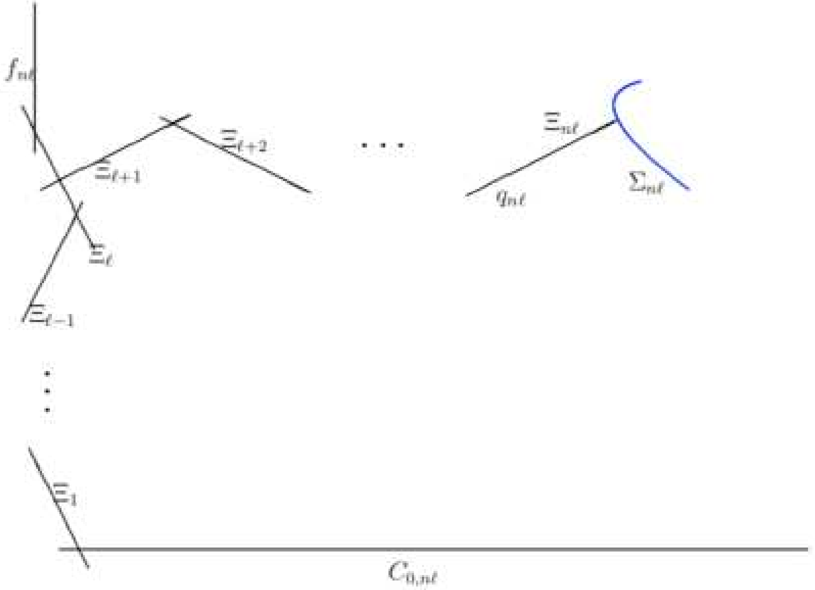
Spectral correspondence

Construction:

- The blow-ups are governed by two mechanisms:
  - untwisting: perform first a sequence of $\ell - 1$ blow-ups to resolve the ℓ-th order tangency between M_p and the spectral curve of type $(n, \ell, 1)$ in M;
  - order of pole reduction: perform a sequence of $(n-1)\ell + 1$ blow-ups to resolve the eigenvalues of the untwisted Laurent tail.
- The canonical class of $M_{n\ell}$ is given by

$$
\begin{aligned}
K_{M_{n\ell}} = & - nf_{n\ell} - (n-1)\Xi_1 - 2(n-1)\Xi_2 - \cdots - \ell(n-1)\Xi_\ell \\
& - (\ell(n-1)-1)\Xi_{\ell+1} - \cdots - \Xi_{n\ell-1} - 0 \cdot \Xi_{n\ell}.
\end{aligned}
$$

Spectral correspondence



Spectral correspondence

Construction:

- $S = M_{n\ell} \backslash (\text{the support of anti-canonical divisor} = f_{n\ell} \cup \bigcup_{i=1}^{n\ell-1} \Xi_i)$
- S is a holomorphic symplectic surface
- there is a unique compact curve $\Sigma_{n\ell} \subset S$ of degree 1 over C. Every compact curve in S is linearly equivalent to $r\Sigma_{n\ell}$ for some $r \geq 1$.
- moduli space $\mathcal{M}_{n,\ell,r}(S)$ of pure dimension one sheaves F with

$$\mathrm{ch}_1(F) = r[\Sigma_{n\ell}] \ , \quad \chi(F) = c = 1 - N(g-1)$$

(with respect to a suitably chosen Bridgeland β-stability)

Theorem (Twisted spectral correspondence)

$$\mathcal{H}_{n,\ell,r} \cong \mathcal{M}_{n,\ell,r}(S)$$

Steps in the strategy

Twisted wild character variety $\mathcal{S}_{n,\ell,r}$	Weighted Poincaré polynomials
↓ NAH, and $P = W$	
Moduli space $\mathcal{H}_{n,\ell,r}$ of twisted irregular Higgs bundles	Perverse Poincaré polynomials
↓ Spectral correspondence	
Moduli space of pure sheaves F of dimension one on spectral surface S	
↓ Gopakumar-Vafa expansion	
Refined stable pairs $\mathcal{O}_Y \xrightarrow{s} F$ of CY3	Refined PT invariants
↓ Large N duality	
Refined Chern-Simons theory of torus knot	Refined Chern-Simons invariants

CY3 and refined invariants

- $Y = K_S = S \times \mathbb{C}^1$

$$\mathcal{M}_{n,\ell,r}(Y) \simeq \mathcal{H}_{n,\ell,r} \times \mathbb{C}^1$$

- Stable pair theory [Pandharipande,Thomas]
$PT_Y(r,c) =$ virtual number of pairs $s : \mathcal{O}_Y \to F$ generically surjective with compact support F and

$$\mathrm{ch}_1(F) = r[\Sigma_{n\ell}], \qquad \chi(F) = c = 1 - N(g-1)$$

- Refined stable pair theory [Kontsevich,Soibelman]
$Z_{PT}^{\mathrm{ref}}(Y; q, y, x) =$ virtual Poincaré polynomial of moduli space of such pairs.

CY3 and refined invariants

Generating functions

$$Z_{PT}^{\mathrm{ref}}(Y; q, y, \mathrm{x}) = 1 + \sum_{r \geq 1,\, c \in \mathbb{Z}} PT_Y^{\mathrm{ref}}(r, c\,; y) q^c \mathrm{x}^r$$

Goal: compute the generating function.

- Specialize $C = \mathbb{P}^1$ with one marked point $\sigma_{n\ell} \in C$
- Torus action $\mathbb{C}^\times \times S \to S$
- Refined virtual localization [Nekrasov,Okounokov][Maulik]
- Stable pair theory localizes on a single rational curve $\Sigma_{n\ell}$ in S

Stable pairs conjecture

Conjecture (Spectral correspondence and Gopakumar-Vafa expansion)

$$\ln Z^{\mathrm{ref}}_{PT}(Y; q, y, \mathrm{x}) = -\sum_{s\geq 1}\sum_{r\geq 1}\frac{\mathrm{x}^{sr}}{s}\frac{y^{sr\#}(qy^{-1})^{s\#}PP_{n,\ell,r}(q^{-s}y^{-s}, y^{s})}{(1-q^{-s}y^{-s})(1-q^{s}y^{-s})}$$

where $PP_{n,\ell,r}$ are perverse Poincaré polynomials

$$PP_{n,\ell,r}(u,v) = \sum_{j,k} u^i v^j \dim Gr^P_i H^j(\mathcal{H}_{n,\ell,r}) \ .$$

Note: Direct localization computations of $PT^{\mathrm{ref}}_{Y}(r, c\, ; y)$ are possible in principle but are hard

Steps in the strategy

Twisted wild character variety $\mathcal{S}_{n,\ell,r}$ — Weighted Poincaré polynomials

NAH, and $P = W$

Moduli space $\mathcal{H}_{n,\ell,r}$ of twisted irregular Higgs bundles — Perverse Poincaré polynomials

Spectral correspondence

Moduli space of pure sheaves F of dimension one on spectral surface S

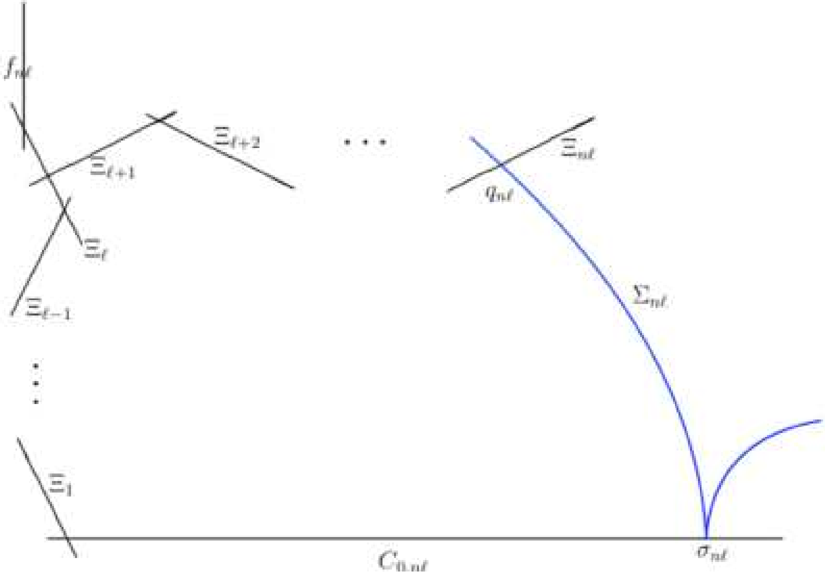
Gopakumar-Vafa expansion

Refined stable pairs $\mathcal{O}_Y \xrightarrow{s} F$ of CY3 — Refined PT invariants

Large N duality

Refined Chern-Simons theory of torus knot — Refined Chern-Simons invariants

Compact curve $\Sigma_{n\ell}$ on S



Refined invariants and refined Chern-Simons theory

- Local equation of the torus invariant curve $\Sigma_{n\ell}$ at $\sigma_{n\ell}$. The curve has a unique singular point of the form

$$v^\ell = w^{(n-2)\ell-1}.$$

- We can use large N duality [Oblomkov,Shende,Rasmussen] [Diaconescu,Hua,Soibelman]

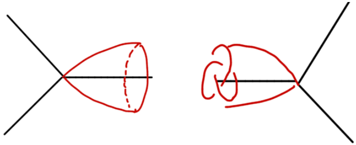
$$\left(\begin{array}{c} \text{Refined PT invariants of} \\ \text{the singular plane curves } \Sigma_{n\ell} \end{array} \right) \leftrightarrow \left(\begin{array}{c} \text{Refined CS invariants of} \\ T_{\ell,(n-2)\ell-1} \text{ torus knots} \end{array} \right)$$

- In refined Chern-Simons theory [Aganagic,Shakirov], torus knot invariants can be evaluated by modular matrices

$$S_{\lambda\mu} = P_\lambda(t^\rho s^\mu) P_\mu(t^\rho) \,, \qquad T_{\lambda\mu} = \delta_{\lambda\mu} s^{\frac{1}{2}\sum_i \lambda_i(\lambda_i-1)} t^{\sum_i \lambda_i(i-1)}$$

Refined invariants and refined Chern-Simons theory

Topological vertex formalism tells us that the contributions are localized at $\sigma_{n\ell}$ and $q_{n\ell}$ of $\Sigma_{n\ell}$



Conjecture (Large N duality)

The generating function of refined stable pairs in Y is expressed as

$$Z^{\mathrm{ref}}_{PT}(Y; q, y, \mathrm{x}) = 1 + \sum_{|\lambda|>0} T^{\#}_{\lambda\lambda} W^{\mathrm{ref}}_{\lambda}(\bigcirc) W^{\mathrm{ref}}_{\lambda}(T_{\ell,(n-2)\ell-1}) \mathrm{x}^{|\lambda|} \Big|_{s=qy,\ t=qy^{-1}}.$$

Epilogue

Definition: the HMW partition function is the function

$$Z_{HMW}(z,w) = 1 + \sum_{|\lambda|>0} \Omega_\lambda^{g,n}(z,w)\widetilde{H}_\lambda(\mathrm{x}; z^2, w^2)$$

- the sum over all Young diagrams λ
- for each such λ

$$\Omega_\lambda^{g,n} = \prod_{\square\in\lambda} \frac{(-z^{2a(\square)}w^{2l(\square)})^{m(n-2)}(z^{2a(\square)+1} - w^{2l(\square)+1})^{2g}}{(z^{2a(\square)+2} - w^{2l(\square)})(z^{2a(\square)} - w^{2l(\square)+2})},$$

- the infinite set of variables $\mathrm{x} = (x_1, x_2, \ldots)$.
- $\widetilde{H}_\lambda(\mathrm{x}; z^2, w^2)$ is the modified Macdonald polynomial

Epilogue

Next let $\mathbb{H}_{g,m,n}(z, w; \mathrm{x}_{(1)}, \ldots, \mathrm{x}_{(m)})$ be defined by

$$\ln Z_{HMW}(z, w) = \sum_{k \geq 1} \frac{(-1)^{(mn-1)|\mu|} \mathbb{H}_{g,m,n}(z^k, w^k; \mathrm{x}^k))}{(1 - z^{2k})(w^{2k} - 1)}.$$

For each Young diagram ρ, let $h_\rho(\mathrm{x})$ and $p_\rho(\mathrm{x})$ be the complete and power sum symmetric functions respectively.

Conjecture

$$WP_{g,m,n,\ell}(u, -1)$$
$$= u^{\#} \left\langle \mathbb{H}_{g,m,n}(u^{\frac{1}{2}}, u^{-\frac{1}{2}}; \mathrm{x}_{(1)}, \ldots, \mathrm{x}_{(m)}) \, , \, \bigotimes_{a=1}^{m} h_{(\ell-1,1)}(\mathrm{x}_{(2a-1)}) \otimes p_{(\ell)}(\mathrm{x}_{(2a)}) \right\rangle$$

where $\langle \, , \, \rangle$ is the natural pairing on symmetric functions.