Actions on topological manifolds: finite abelian groups vs tori

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A general pattern in rigidity resuts

$\mathcal{S} = \{\text{geometric objects of some type}\}$, functional $\phi : \mathcal{S} \longrightarrow \mathbb{R}$

Typical rigidity phenomenon

- There is an upper bound $\phi(\mathcal{S}) \subset (-\infty, M]$
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- $U(p, q)$-local systems on surfaces after Toledo, Hernández, and Bradlow-García–Prada-Gothen (ϕ =Toledo invariant)

A rigidity result on actions of tori

$X =$ connected topological manifold

An action of a group G on X is a morphism $\phi : G \to \mathrm{Homeo}(X)$, which is continuous if G is a topological group. Notation: $G \curvearrowright X$. The action is *effective* if ϕ is injective.

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Theorem

Let $n = \dim X$. We have $A(X) \leqslant n$, and $A(X) = n$ if and only if $X \cong T^n$.

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Proof. Exercise. (Hint: use the theorem on invariance of domain: if $U \subset \mathbb{R}^n$ is open and $\phi : U \to \mathbb{R}^n$ is continuous and injective, then $\phi(U) \subset \mathbb{R}^n$ is open.)

Actions of finite abelian groups

$X =$ closed connected topological manifold

Theorem (Mann-Su, 1965)

There exists some number $R(X)$ such that, for any prime p, if there is an effective action of $(\mathbb{Z}/p)^m$ on X then $m \leqslant R(X)$

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Consequently, the following is a well defined nonnegative integer

$$D(X) = \max\{m \mid \exists \text{ integers } r_i \to \infty,\ \forall i\ \exists\ (\mathbb{Z}/r_i)^m \hookrightarrow \mathrm{Homeo}(X)\}$$

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Lemma

The following are equivalent:

- $D(X) \leqslant k$,
- there exists a number C, depending only on X, such that every finite abelian group A acting effectively on X has a subgroup $A' \leqslant A$ such that $[A : A'] \leqslant C$ and such that A' can be generated by k or fewer elements.

Sequences $(\mathbb{Z}/r_i)^m$ with $r_i \to \infty$ approximate $(S^1)^m$

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Let X be a compact Kaehler manifold. Define

$$A_{\text{hol}}(X) = \max\{m \mid \exists\, T^m \hookrightarrow \text{Bihol}(X)\},$$

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Theorem (M., 2021)

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However, we could still ask:

Question

- Is $D(X) \leqslant \dim X$ for every closed conn.'d top. manifold X?
- If X is a closed conn.'d topological manifold satisfying $D(X) = \dim X$, is X homeomorphic to a torus?

Comments on the Question

- For a fixed closed connected topological manifold X we consider a **sequence** of finite abelian groups $\{(\mathbb{Z}/r_i)^m\}_i$ acting effectively on X. If we assume that X supports an effective action of a single group $(\mathbb{Z}/r)^m$, then there is no reason to expect that $m \leqslant \dim X$ unless r is sufficiently big (depending on X). **Example:** any finite group G acts freely on some closed connected surface.

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- We may ask a similar question for actions by birational transformations on algebraic varieties. In this context, the answer to the question is... **YES**!

Theorem (A. Golota, 2022 — see `arXiv:2205.00607`)

Let X be a projective variety over an algebraically closed field of zero characteristic. Suppose that $\mathrm{Bir}(X)$ contains subgroups isomorphic to $(\mathbb{Z}/r_i)^m$ for arbitrarily large r_i. Then $r \leqslant 2\dim(X)$ and in case of equality X is birational to an abelian variety.

Positive results

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A closed connected $2d$-dimensional manifold X is c-**symplectic of Lefschetz type** if $\exists\, \Omega \in H^2(X;\mathbb{R})$ such that $\Omega^d \neq 0$ and such that $\cdot \cup \Omega^{d-1} : H^1(X;\mathbb{R}) \to H^{2d-1}(X;\mathbb{R})$ is an isomorphism. **Example:** compact Kaehler manifolds are c-symplectic of Lefschetz type.

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Theorem (M. 2021/22 — see `arXiv:2112.05599`)

Let X be a closed connected n-dimensional manifold. Suppose that X is *rationally hypertoral* or *c-symplectic of Lefschetz type*.

- We have $D(X) \leqslant n$.
- If $D(X) = n$ then $H^*(X;\mathbb{Z}) \simeq H^*(T^n;\mathbb{Z})$, and if in addition $\pi_1(X)$ is virtually solvable then X is homeomorphic to T^n.

Examples of computations of $D(X)$

Surfaces

- $D(S^2) = D(\mathbb{R}P^2) = D(\text{Klein bottle}) = 1$
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Some higher dimensional manifolds

- Let Y be a closed connected d-dimensional manifold with $H_*(Y;\mathbb{Z}) \not\cong H_*(S^d;\mathbb{Z})$. Let $X = T^m \times (T^d \sharp Y)$. There is a map $X \to T^{m+d}$ of degree 1, and $D(X) = m$.

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- Let $k, n \in \mathbb{N}$ satisfy $1 \leqslant k \leqslant n-1$. Define $\sigma : T^n \to T^n$ by $\sigma(x_1, \ldots, x_n) = (x_1 + 1/2, \ldots, x_k + 1/2, -x_{k+1}, \ldots, -x_n)$. (We identify $T^n = (\mathbb{R}/\mathbb{Z})^n$.) We have $D(T^n/\sigma) = k$.

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In all these examples $A(X) = D(X)$.

Some simplifications

Lemma

Let X be a closed connected manifold satisfying $D(X) = m$. At least one of the following statements is true.

- $\exists\ (\mathbb{Z}/p)^m \hookrightarrow \mathrm{Homeo}(X)$ for arbitrarily big primes p,
- for some fixed prime p, $\exists\ (\mathbb{Z}/p^e)^m \hookrightarrow \mathrm{Homeo}(X)$ for all e.

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For simplicity we will sketch the proof of the following.

Theorem

Let X be a closed conn'd n-dimensional manifold. Suppose that

- X is c-symplectic of Lefschetz type,
- $\exists\ (\mathbb{Z}/p)^m \hookrightarrow \mathrm{Homeo}(X)$ for arbitrarily big primes p.

Then $m \leqslant n$. If $m = n$ then $H_*(X;\mathbb{Z}) \simeq H_*(T^n;\mathbb{Z})$. If $m = n$ and $\pi_1(X)$ is virtually solvable, then $X \cong T^n$.

The case $b_1(X) = 0$, part 1/3

Lemma

Let X be a closed connected $2d$-dim'l manifold with $b_1(X) = 0$. Suppose $\exists\, \Omega \in H^2(X; \mathbb{R})$ such that $\Omega^d \neq 0$. Then $\exists\, p_0$ such that, $\forall$ prime $p \geqslant p_0$, every $\mathbb{Z}/p \circlearrowright X$ has a fixed point.

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- $\exists\, \lambda \in H^2(X;\mathbb{Z})$ s.t. Then $\lambda^d = a[X]$, where $[X]$ generates $H^{2d}(X;\mathbb{Z})$ and $0 \neq a \in \mathbb{N}$.

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- $\exists\, \lambda \in H^2(X;\mathbb{Z})$ s.t. Then $\lambda^d = a[X]$, where $[X]$ generates $H^{2d}(X;\mathbb{Z})$ and $0 \neq a \in \mathbb{N}$.
- Minkowski: $\exists\, b \in \mathbb{N}$ s.t. $\forall$ prime $p \geqslant b$, any $\mathbb{Z}/p \circlearrowright X$ induces trivial action on $H^*(X,\mathbb{Z})$. Let $p_0 = \max\{a+1, b\}$.

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- Let $p \geqslant p_0$ be a prime, and assume $G = \mathbb{Z}/p \circlearrowright X$. Borel fib.: $X \hookrightarrow X_G = EG \times_G X \to BG$. Serre s.s.: $H^p(BG; H^q(X;\mathbb{Z}))$ $\Rightarrow H^{p+q}(X_G;\mathbb{Z})$. Then λ extends to $H^2(X_G;\mathbb{Z})$ since $H^2(BG; H^1(X;\mathbb{Z})) = H^3(BG; H^0(X;\mathbb{Z})) = 0$.

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- If $G \circlearrowright X$ freely, then $\pi : X \to X/G \simeq X_G$ has deg. p and $\lambda \in \pi^* H^2(X_G;\mathbb{Z})$, so $\lambda^d \in \pi^* H^{2d}(X_G;\mathbb{Z})$, so $p \mid a \leqslant p_0$!!!

The case $b_1(X) = 0$, part 2/3

Theorem (M., 2022)

Let X be a closed connected manifold. $\exists\, C = C(X)$ satisfying: let G be a finite group, assume $G \circlearrowright X$ effectively, and that, $\forall\, g \in G$, $\exists\, x \in X$ such that $g \cdot x = x$; then $\exists\, H \leqslant G$ such that: $[G : H] \leqslant C$, H is abelian, and $X^H \neq \emptyset$.

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Let X be a closed connected manifold. $\exists\, C = C(X)$ satisfying: let G be a finite group, assume $G \circlearrowright X$ effectively, and that, $\forall\, g \in G$, $\exists\, x \in X$ such that $g \cdot x = x$; then $\exists\, H \leqslant G$ such that: $[G : H] \leqslant C$, H is abelian, and $X^H \neq \emptyset$.

Some applications:

- For any $n \in \mathbb{N}$, $\exists\, C(n)$ s.t. for any finite group G and any $G \circlearrowright D^n$, there is a point $x \in D^n$ s.t. $[G : G_x] \leqslant C(n)$ (use Brouwer's fixed point theorem).

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- X c-symplectic manifold of Lefschetz type. $\exists\, p_1$ such that for every prime $p \geqslant p_1$ any $(\mathbb{Z}/p)^m \circlearrowright X$ satisfies $X^{(\mathbb{Z}/p)^m} \neq \emptyset$ (use previous Lemma).

The case $b_1(X) = 0$, part 3/3

Lemma (based on Dotzel-Hamrick, 1981)

Let X be a connected n-dimensional topological manifold and G a finite p-group. Suppose that $G \circlearrowright X$ effectively and $X^G \neq \emptyset$. Then G is isomorphic to a subgroup of $\mathrm{GL}(n, \mathbb{R})$.

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For odd primes p, $(\mathbb{Z}/p)^m \leqslant \mathrm{GL}(n, \mathbb{R})$ implies $m \leqslant [n/2]$. Putting everything together we obtain:

Theorem

Let X be a c-symplectic manifold of Lefschetz type of dimension $2d$. If $b_1(X) = 0$ then there exists some p_2 such that for every prime $p \geqslant p_2$ and any effective $(\mathbb{Z}/p)^m \curvearrowright X$ we have $m \leqslant d$.

Abelian covers

Given a map $\phi : X \to T^k$ define X_ϕ through the Cartesian diagram

$$\begin{array}{ccc} X_\phi & \longrightarrow & \mathbb{R}^k \\ \downarrow & & \downarrow \pi \\ X & \xrightarrow{\phi} & T^k = \mathbb{R}^k/\mathbb{Z}^k. \end{array}$$

$\mathbb{Z}^k \circlearrowright X_\phi \Rightarrow H_*(X_\phi; \mathbb{Z})$ is a module over $\mathbb{Z}[\mathbb{Z}^k] \simeq \mathbb{Z}[z_1^{\pm 1}, \ldots, z_k^{\pm 1}]$.

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Suppose that X is a closed manifold. Then:

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Rotation morphism

X closed connected manifold, $\alpha_1, \ldots, \alpha_k$ basis of $H^1(X;\mathbb{Z})$.
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$\exists$ morphisms $\rho_j : G \to S^1$ defined by $\rho_j(g) = \phi_j(g \cdot x) - \phi_j(x)$ for any x. Then $\rho = (\rho_1, \ldots, \rho_k)$ is the **rotation morphism**.

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$$\begin{array}{ccc} G \times X & \longrightarrow & X \\ \downarrow{\scriptstyle \rho\times\phi} & & \downarrow{\scriptstyle \phi=(\phi_1,\ldots,\phi_k)} \\ T^k \times T^k & \longrightarrow & T^k. \end{array}$$

The action of $\operatorname{Ker}\rho$ on X lifts to an action on X_ϕ.

Proof of the Theorem: general case

Suppose X is a c-symplectic manifold of Lefschetz type, $p_i \to \infty$ are primes, and $G_i := (\mathbb{Z}/p_i)^m \circlearrowright X$ effectively. Suppose $H^1(X;\mathbb{Z}) \simeq \mathbb{Z}^k$.

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Theorem

$H_*(X_{\phi_J};\mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module.

Consequence: $b \leqslant \dim X =: 2d$.

Why is $H_*(X_{\phi_J}; \mathbb{Z})$ finitely generated?

Theorem (M. 2021)

Let A be a Noetherian ring, M a f.g. $A[z]$-module. Suppose there exists some nonzero $\delta \in \mathbb{N}$, integers $r_i \to \infty$, and $w_i \in \mathrm{End}_{A[z]} M$ such that $w_i^{r_i} = z^\delta \ \forall i$. Then M is a f.g. A-module.

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Apply the theorem to $M = H_*(X_{\phi_J};\mathbb{Z})$. For each $1 \leqslant j \leqslant b$, $w_{j,i}$ is the action on M of any $g_{i,j} \in \rho_J^{-1}(0, \dots, [1/p], \dots, 0) \subset G_i$, where $[1/p]$ is in the j-th position. Use an analogue of Minkowski lemma for finite subgroups of $\mathrm{Aut}_B M$ to get δ.

The kernel of the rotation morphism

Let $G_{i,f} = \mathrm{Ker}(\rho_J : G_i \to T^b) \simeq (\mathbb{Z}/p_i)^{m-b}$. We have an exact sequence

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$$m = b + (m - b) \leqslant d + b/2 \leqslant 2d = \dim X.$$

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F_i is a $\mathbb{Z}/p_i$-cohomology manifold, and F_i lifts to the $\mathbb{Z}^b$-invariant cohomology manifold $\widetilde{F}_i = (X_{\phi_J})^{G_{i,f}}$. Since $H_*(X_{\phi_J}; \mathbb{Z}/p_i)$ is finite, so is $H_*(\widetilde{F}_i; \mathbb{Z}/p_i)$ (Smith theory). Applying the Serre s.s.:

$$\dim F_i \geqslant b.$$

Arguing as in the case $b_1(X) = 0$ we conclude that $G_{i,f} \hookrightarrow \mathrm{GL}(2d - b; \mathbb{R})$. Hence $m - b \leqslant d - b/2$. Then

$$m = b + (m - b) \leqslant d + b/2 \leqslant 2d = \dim X.$$

Equality only if $b = 2d$. Then $H_*(X_{\phi_J}; \mathbb{Z}) \simeq H_*(\mathrm{pt}; \mathbb{Z})$, so $H_*(X; \mathbb{Z}) \simeq H_*(T^{2d}; \mathbb{Z})$.

Conclusion of the proof

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Equality only if $b = 2d$. Then $H_*(X_{\phi_J}; \mathbb{Z}) \simeq H_*(\mathrm{pt}; \mathbb{Z})$, so $H_*(X; \mathbb{Z}) \simeq H_*(T^{2d}; \mathbb{Z})$. If $\pi_1(X)$ virtually abelian, X_{ϕ_J} is contractible $\Rightarrow X \simeq T^{2d} \Rightarrow X \cong T^{2d}$ (topological rigidity of tori).

THANK YOU

AND

HAPPY BIRTHDAY, ÓSCAR!