Higgs fields in low genus

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STABLE BUNDLES

- curve C $g > 1$

  rank n stable bundle E, fixed $\Lambda^n E$

- moduli space $\mathcal{N}$, dimension $(n^2-1)(g-1)$

  tangent space $T_E \cong H^1(C, \mathsf{End}_0\, E)$

- Serre duality: cotangent space $H^0(C, \mathsf{End}_0\, E \otimes K)$

  $=$ Higgs fields on E

CLASSICAL CONSTRUCTIONS

- $g = 2$ rank 2 $\Lambda^2 E$ trivial, $\mathcal{N} \cong \mathsf{P}^3$

- $g = 2$ rank 2 odd degree

  $\mathcal{N} \cong$ intersection of two quadrics in P^5

M.S.Narasimhan & S.Ramanan, *Moduli of vector bundles on a compact Riemann surface*, Ann. of Math. **89** (1969) 19–51.

P.Newstead, *Stable bundles of rank 2 and odd degree over a curve of genus* 2, Topology **7** (1968) 205–215.

- $g = 3$ rank 2 $\Lambda^2 E$ trivial

  $\mathcal{N} \cong$ Coble quartic hypersurface in P^7

  M.S.Narasimhan & S.Ramanan, 2Θ *systems on abelian varieties*, Vector bundles and algebraic varieties, Bombay 1984, OUP (1987) 415 – 427.

- $g = 2$ rank 3 $\Lambda^3 E$ trivial

  $\mathcal{N} \cong$ double cover of P^8 branched over Coble sextic

  A. Ortega, *On the moduli space of rank 3 vector bundles on a genus 2 curve and the Coble cubic*, J. Algebraic Geom. **14** (2005) 327-356

- A.Beauville, *The Coble hypersurfaces*, C.R.Acad.Sci.Paris **337** (2003) 189–194.

- finite Heisenberg group $1 \to \mu_n \to H_n \to A_n = H^1(C, \mathbf{Z}_n) \to 0$

  H_n acts on vector space V, A_n on $\mathsf{P}(V)$

- $E \mapsto E \otimes L$, L^n trivial

  $\Rightarrow$ action of $H^1(C, \mathbf{Z}_n)$ on $\mathcal{N}$ (e.g. $\mathsf{P}^3, \mathsf{P}^7, \mathsf{P}^8$)

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  $\Rightarrow$ action of $H^1(C, \mathbf{Z}_n)$ on $\mathcal{N}$ (e.g. $\mathsf{P}^3, \mathsf{P}^7, \mathsf{P}^8$)

- $\Phi \in H^0(C, \mathsf{End}_0\, E \otimes K)$ independent of action

- $\Gamma = H^1(C, \mathbf{Z}_2)$

- genus 2 even degree $\mathcal{N} = \mathsf{P}^3$

  $\mathcal{N}/\Gamma =$ Igusa quartic $\subset \mathsf{P}^4$

- genus 2 odd degree

  $\mathcal{N}/\Gamma =$ double covering of P^3 branched over six planes

  M.F.Atiyah, *Complex fibre bundles and ruled surfaces*, Proc. Lond. Math. Soc. **5** (1955) 407–434.

TWO ISSUES

NJH, *Stable bundles and integrable systems*, Duke Math.J. **54** (1987), 91–114.

- rank 2:

  $\mathrm{tr}\,\Phi^2 : H^0(C, \mathrm{End}_0\, E \otimes K) \to H^0(C, K^2)$

- rank 3:

  $(\mathrm{tr}\,\Phi^2, \mathrm{tr}\,\Phi^3) : H^0(C, \mathrm{End}_0\, E \otimes K) \to H^0(C, K^2) \oplus H^0(C, K^3)$

- fixed E very stable (no nilpotent Higgs fields):

  proper map $h : \mathbf{C}^N \to \mathbf{C}^N \sim$ "multiplicity algebra" (Hausel)

NJH, *Stable bundles and integrable systems*, Duke Math.J. **54** (1987), 91–114.

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- vary E, holomorphic sections of S^2T, S^3T on $\mathcal{N}$

NJH, *Stable bundles and polyvector fields*, in "Complex and Differential Geometry", W. Ebeling et al (eds.) Springer Proceedings in Mathematics **8**, 135–156, Springer Verlag, Heidelberg (2011).

- $\mathrm{tr}(\Phi_1[\Phi_2, \Phi_3]) : \Lambda^3 H^0(C, \mathrm{End}_0\, E \otimes K) \to H^0(C, K^3)$

- vary E, holomorphic sections of $\Lambda^3 T$ on $\mathcal{N}$

TWO KEY PROPERTIES

Heisenberg group acts trivially on $\mathrm{End}_0\, E$

$\Rightarrow$ tensors $S^m T, \Lambda^n T$ on $\mathcal{N}$ from invariant forms

are Heisenberg-invariant

- holomorphic sections of $\Lambda^k T$ on M

- Schouten bracket α degree k, β degree ℓ

  $[\alpha, \beta] \in H^0(M, \Lambda^{k+\ell-1} T)$

  - $k = \ell = 1$ Lie bracket $[X, Y]$

    $k = 2$, $[\alpha, \alpha] = 0$ Poisson structure

- **Prop**: image of $H^1(C, K^{-2})$ commutative

RANK 2, GENUS 2, ODD DEGREE

M.F.Atiyah, *Complex fibre bundles and ruled surfaces*, Proc. Lond. Math. Soc. **5** (1955) 407–434.

- $\mathrm{End}_0\, E$ independent of $E \mapsto E \otimes L$

  assume extension $\mathcal{O} \to E \to L$, $\deg L = 1$

- extension class $[\alpha] \in H^1(C, L^*)$ $\dim H^1(C, L^*) = 2$

  ... determined by $s \in H^0(C, KL)$ with $[\alpha]s = 0 \in H^1(C, K)$

- divisor of s: $p + q + r$

- E has 4 degree zero subbundles

- hyperelliptic involution $\sigma : C \to C$, $\pi : C \to \mathsf{P}^1$

- divisors:

  $p + q + r$

$$p + \sigma(q) + \sigma(r), \quad q + \sigma(r) + \sigma(p), \quad r + \sigma(p) + \sigma(q)$$

- E has 4 degree zero subbundles

- hyperelliptic involution $\sigma : C \to C$, $\pi : C \to \mathsf{P}^1$
  branch points $x_1, \ldots, x_6 \in \mathsf{P}^1$
- divisors:

  $p + q + r$

  $p + \sigma(q) + \sigma(r), \quad q + \sigma(r) + \sigma(p), \quad r + \sigma(p) + \sigma(q)$

- $(\pi(p), \pi(q), \pi(r)) \in S^3(\mathsf{P}^1) = \mathsf{P}^3$

  moduli space double cover =

  cubic polynomials $p(z)$, branched over $p(x_i) = 0, i = 1, \ldots, 6$

- $V = \{p(x) = ax^3 + bx^2 + cx + d\}$, $\mathsf{P}^3 = \mathsf{P}(V)$

- discriminant sextic surface Δ

$$b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd = 0$$

- $f : \mathcal{N} \to \mathsf{P}^3$, wobbly locus $= f^{-1}(\Delta)$

NJH, *Spinors, twistors and classical geometry*, SIGMA **17** (2021)

$$C : y^2 = (x - x_1)(x - x_2) \dots (x - x_6) \qquad \sigma(x, y) = (x, -y)$$

- C^∞ splitting $E = \mathcal{O} \oplus L$

  extension defined by $\alpha = \bar{\partial} u / s \in \Omega^{01}(C, L^*) \Rightarrow s[\alpha] = 0$

- E very stable (no nilpotent Higgs fields)

  can choose u supported in a nhd of p

Higgs fields:

$$\Phi_1 = (x-a_2)\begin{pmatrix} 1 & 2s^{-1}(u - u_1\frac{(x-a_3)}{(a_1-a_3)}) \\ 0 & -1 \end{pmatrix}\frac{dx}{y}$$

$$\Phi_2 = (x-a_3)\begin{pmatrix} 1 & 2s^{-1}(u - u_1\frac{(x-a_2)}{(a_1-a_2)}) \\ 0 & -1 \end{pmatrix}\frac{dx}{y}$$

$$\Phi_3 = \begin{pmatrix} -u & s^{-1}(-u^2 + u_1^2\frac{(x-a_2)(x-a_3)}{(a_1-a_2)(a_1-a_3)}) \\ s & u \end{pmatrix}\frac{dx}{y}$$

$$(\pi(p,q,r) = (a_1,a_2,a_3),\ u(p) = u_1)$$

- $\mathcal{O} \subset E$ preserved by all Higgs fields at $p, q, r \in C$

- Hecke transform to $\mathcal{O} \oplus K$

- $\Rightarrow$ multiplicity algebra $\cong H^*((\mathrm{P}^1)^3, \mathbf{C})$

symmetric tensor

- $H^1(C, K^{-1}) \to H^0(\mathcal{N}, S^2T)$

- $f(a_1)\dfrac{(x-a_2)(x-a_3)}{(a_1-a_2)(a_1-a_3)}\left(\dfrac{\partial}{\partial a_1}\right)^2 + f(a_2)\dfrac{(x-a_3)(x-a_1)}{(a_2-a_3)(a_2-a_1)}\left(\dfrac{\partial}{\partial a_2}\right)^2 +$

$$+ f(a_3)\frac{(x-a_1)(x-a_2)}{(a_3-a_1)(a_3-a_2)}\left(\frac{\partial}{\partial a_3}\right)^2 \times \frac{dx^2}{y^2}$$

skew-symmetric tensor

- $H^1(C, K^{-2}) \to H^0(\mathcal{N}, \Lambda^3 T)$

- $\rho = y_1 y_2 y_3 \dfrac{(x-a_1)(x-a_2)(x-a_3)}{(a_2-a_3)(a_3-a_1)(a_1-a_2)} \dfrac{\partial}{\partial a_1} \wedge \dfrac{\partial}{\partial a_2} \wedge \dfrac{\partial}{\partial a_3}$

$$(y_1, y_2, y_3) = (y(p), y(q), y(r))$$

$$H^0(C, K^3) : (c_0 y + c_1 + c_2 x + c_3 x^2 + c_4 x^3) \frac{dx^3}{y^3}$$

ρ anti-invariant by $\sigma : C \to C$

- $p : \mathcal{N} \to \mathsf{P}^3$ double covering, branched over sextic

  P^3 = quotient by action of σ

  $p_*\mathcal{O} \cong \mathcal{O} \oplus \mathcal{O}(-3)$ ⟵ anti-invariant

- local section of $\Lambda^3 T^*_{\mathsf{P}3}$, pull back to $\mathcal{N}$

  ... evaluate ρ, direct image = section of $\Lambda^3 T_{\mathsf{P}3} \oplus \Lambda^3 T_{\mathsf{P}3}(-3)$

- $\mathcal{O}(-1) \to \mathcal{O}^4 \to T_{\mathsf{P}3}(-1) \Rightarrow H^0(\mathsf{P}^3, \Lambda^3 T_{\mathsf{P}3}(-3)) \cong \mathbf{C}^4$

RANK 2, GENUS 2, DEGREE 0

B.van Geemen & E.Previato, *On the Hitchin system*, Duke Math.J. **85** (1996), 659–683.

- $\mathcal{N} = \mathsf{P}^3 = \mathsf{P}(H^0(\mathsf{Pic}^1(C), 2\Theta))$

  $D(E) = \{L \in \mathsf{Pic}^1(C) : \dim H^0(C, E \otimes L) > 0\}$: 2Θ-divisor

- $D(E) =$ genus 5 curve C_5

  $L \mapsto KL^*$ free involution τ

  $C_5 \mapsto C_5/\tau = C_3$ Prym variety $\cong \mathsf{Pic}^1(C)$

  $H^0(C, \mathsf{End}_0\, E \otimes K) \cong H^1(C_3, \mathcal{O})$

K Gawędzki and P Tran-Ngoc-Bich: *Self-duality of the* SL_2 *Hitchin integrable system at genus* 2, Comm. Math. Phys. **196** (1998) 641–670.

- symplectic quotient: $(p,q) \mapsto (\lambda p, \lambda^{-1} q)$

$$T^*\mathsf{P}(V) = \{(p,q) \in V \times V^* : \langle p,q\rangle = 0, p \neq 0\}/\mathbf{C}^*$$

- $$H = \sum_{i \neq j} \frac{r_{ij}(p,q)}{(x-x_i)(x-x_j)} dx^2$$

$$r_{12} = (q_1p_1+q_2p_2-q_3p_3-q_4p_4)^2, r_{13} = (q_1p_4-q_2p_3-q_3p_2+q_4p_1)^2$$

$$r_{14} = -(q_1p_4+q_2p_3-q_3p_2-q_4p_1)^2, r_{15} = -(q_1p_3-q_2p_4-q_3p_1+q_4p_2)^2$$

etc.

- algebra relations $\cong$ quadratic forms on $\mathbf{C}^3$ parametrized by $H^1(C, K^{-1}) \cong \mathbf{C}^3$

  discriminant = cubic curve in $\mathsf{P}^2 = \mathsf{P}(H^1(C, K^{-1}))$

- $K^{1/2}$ odd theta characteristic $\Rightarrow H^0(C, \mathsf{End}_0\, E \otimes K^{1/2}) \neq 0$

  Weierstrass points x_i, $i = 1, \ldots, 6$, $K^{1/2} \sim \mathcal{O}(x_i)$

  $\Rightarrow$ at least one Higgs field vanishes at x_i

  $\Rightarrow$ discriminant passes through each x_i

  $\Rightarrow$ modulus of cubic curve varies, invariant of the algebra

- $\rho \in H^0(\mathcal{N}, \Lambda^3 T)$?

- σ acts trivially on P^3 hence on any ρ from $H^1(C, K^{-2})$

  dx^3/y^2 only invariant section of $K^3 \Rightarrow$ unique ρ

- $\Lambda^3 T \cong \mathcal{O}(4) \Rightarrow$ quartic

  ... vanishes on Kummer surface

  (can evaluate on simple extension $L \to E \to L^*$

- **Thm** (NJH 2009) $H^1(C, K^{-2}) \to H^0(\mathcal{N}, \Lambda^3 T)$ is injective for $g > 2$ for rank 2 bundles.

- irreducible rep $SU(2) \to G \Rightarrow$ injective for G.

GENUS 2, RANK 3

A. Ortega, *On the moduli space of rank 3 vector bundles on a genus 2 curve and the Coble cubic*, J. Algebraic Geom. **14** (2005) 327-356

Q.M.Nguyen, *Vector bundles, dualities and classical geometry on a curve of genus two*, Internat. J. Math. **18** (2007) 535-558

E.Rains & S. Sam, *Invariant theory of $\Lambda^3(9)$ and genus 2 curves*, Algebra Number Theory, **12** (2018) 935–957.

E.Rains & S. Sam, *Vector bundles on genus 2 curves and trivectors*, Algebraic Geometry **6** (2019) 328–345.

- $D(E) = \{L \in \mathsf{Pic}^1(C) : \dim H^0(C, E \otimes L) \neq 0\}$

  3Θ-divisor

- moduli space $\mathcal{N}$ = double covering of $\mathsf{P}^8 = \mathsf{P}(H^0(\mathsf{Pic}^1(C), 3\Theta))$
  branched over a sextic hypersurface C_6

  ... dual to Coble cubic

- involution on $\mathcal{N}$: $\tau^* E = \sigma^* E^*$

- involution $L \mapsto KL^*$

  fixed points in $\mathsf{P}^8 = \mathsf{P}^4_+ \cup \mathsf{P}^3_-$

- P^3_- intersects C_6 in 6 planes

- V rank 2, odd degree $E = \mathsf{End}_0\, V$

  double covering of P^3_- branched over 6 planes

  = Atiyah description

- P^4_+ intersects C_6 in a quartic and a double plane

- quartic = Igusa quartic P^3/Γ =
  bundles $E = \mathsf{End}_0\, V$, $\Lambda^2 V \cong \mathcal{O}$

- $\sigma^* V \cong V \cong V^* \Rightarrow$ contained in C_6

- V odd/even degree $\Rightarrow$ anti-invariant/invariant ρ

- $\Rightarrow H^1(C, K^{-2}) \to H^0(\mathcal{N}, \Lambda^3 T)$ injective

- $\dim \mathcal{N} = 8$, $\rho_i, i = 1, ...5$

$$[\rho_i, \rho_j] = 0 \in H^0(\mathcal{N}, \Lambda^5 T)$$

- $p : \mathcal{N} \to \mathsf{P}^8$ branched over sextic

  $p_* \mathcal{O}_M \cong \mathcal{O} \oplus \mathcal{O}(-3)$

- pull back local section of $\Lambda^3 T^*_{\mathsf{P}^8}$, evaluate ρ, push forward defines a section of $\Lambda^3 T_{\mathsf{P}^8} \oplus \Lambda^3 T_{\mathsf{P}^8}(-3)$

- invariant/anti-invariant?

- involution $\tau : \mathcal{N} \to \mathcal{N}$

  $\tau^* E = \sigma^* E^*$ and $\tau^* \Phi = -\sigma^* \Phi^T$

- $\tau^* \operatorname{tr}(\Phi_1[\Phi_2, \Phi_3]) = -\sigma^* \operatorname{tr}(\Phi_1^T[\Phi_2^T, \Phi_3^T]) = \sigma^* \operatorname{tr}(\Phi_1[\Phi_2, \Phi_3])$

- anti-invariant $H^1(C, K^{-2}) \cong \mathbf{C}^4$

- $H^0(\mathsf{P}^8, \Lambda^3 T(-3)) \cong \Lambda^3 \mathbf{C}^9$

  Heisenberg-invariant subspace $\cong \mathbf{C}^4$

SYMMETRIC TENSORS

- rank 3: $\operatorname{tr}\Phi^2, \operatorname{tr}\Phi^3$

- $\tau^*\Phi = -\sigma^*\Phi^T$

  $\tau^* \operatorname{tr}\Phi^3 = -\sigma^* \operatorname{tr}(\Phi^T)^3 = -\sigma^* \operatorname{tr}\Phi^3$

- evaluate on the invariant element in $H^1(C, K^{-2})$

  $\Rightarrow$ section of $S^3 T_{\mathbb{P}^8}(-3)$

SYMMETRIC TENSORS

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- evaluate on the invariant element in $H^1(C, K^{-2})$

  $\Rightarrow$ section of $S^3 T_{\mathbb{P}^8}(-3)$

- Heisenberg-invariant cubic forms $\cong \mathbf{C}^5$....

- Coble cubic = dual of C_6 defines the section of $S^3 T$

GENUS 3 RANK 2

- $g = 3$ rank 2 $\Lambda^2 E$ trivial

  $\mathcal{N} \cong$ Coble quartic hypersurface in P^7

  M.S.Narasimhan & S.Ramanan, 2Θ *systems on abelian varieties*, Vector bundles and algebraic varieties, Bombay 1984, OUP (1987) 415 – 427.

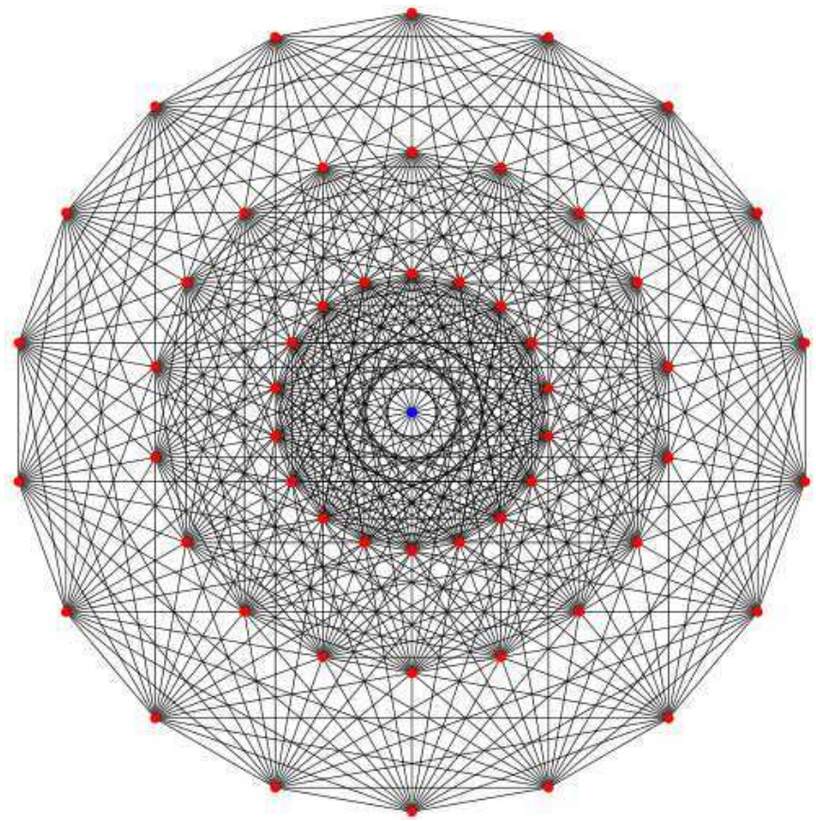
Steven Sam, *Combinatorics and geometry of* E_7, pdf talk 2012

Combinatorics and geometry of E_7

Steven Sam

University of California, Berkeley

September 19, 2012



- involution $\theta : \mathfrak{e}_7 \to \mathfrak{e}_7$

  decomposition $\mathfrak{e}_7 \cong \mathfrak{sl}(8) \oplus \Lambda^4 \mathbf{C}^8$

- Heisenberg group H acts on $\mathbf{C}^8$

  $H \subset GL(8)$, $N(H)/H \cong Sp(6, \mathbf{Z}_2)$

  Weyl group $W(E_7) \cong \mathbf{Z}_2 \times Sp(6, \mathbf{Z}_2)$

$$a_1 = x_{000},\ a_2 = x_{100},\ a_3 = x_{010},\ a_4 = x_{110},$$
$$a_5 = x_{001},\ a_6 = x_{101},\ a_7 = x_{011},\ a_8 = x_{111}.$$

The Heisenberg group H acts on $\bigwedge^4 A$ and the space of invariants has the following basis:

$$\begin{aligned}
h_1 &= a_1 \wedge a_2 \wedge a_3 \wedge a_4 + a_5 \wedge a_6 \wedge a_7 \wedge a_8,\\
h_2 &= a_1 \wedge a_2 \wedge a_5 \wedge a_6 + a_3 \wedge a_4 \wedge a_7 \wedge a_8,\\
h_3 &= a_1 \wedge a_3 \wedge a_5 \wedge a_7 + a_2 \wedge a_4 \wedge a_6 \wedge a_8,\\
h_4 &= a_1 \wedge a_4 \wedge a_6 \wedge a_7 + a_2 \wedge a_3 \wedge a_5 \wedge a_8,\\
h_5 &= a_1 \wedge a_3 \wedge a_6 \wedge a_8 + a_2 \wedge a_4 \wedge a_5 \wedge a_7,\\
h_6 &= a_1 \wedge a_4 \wedge a_5 \wedge a_8 + a_2 \wedge a_3 \wedge a_6 \wedge a_7,\\
h_7 &= a_1 \wedge a_2 \wedge a_7 \wedge a_8 + a_3 \wedge a_4 \wedge a_5 \wedge a_6.
\end{aligned}$$

- Coble quartic $Q \subset \mathsf{P}^7$

- $0 \to \mathcal{O}_{\mathsf{P}^7}(-1) \to \mathbf{C}^8 \to T_{\mathsf{P}^7}(-1) \to 0$

  $0 \to \Lambda^3 T_{\mathsf{P}^7}(-4) \to \Lambda^4 \mathbf{C}^8 \to \Lambda^4 T_{\mathsf{P}^7}(-4) \to 0$

  $\Rightarrow H^0(\mathsf{P}^7, \Lambda^4 T_{\mathsf{P}^7}(-4)) \cong \Lambda^4 \mathbf{C}^8$

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  $\Rightarrow H^0(\mathsf{P}^7, \Lambda^4 T_{\mathsf{P}^7}(-4)) \cong \Lambda^4 \mathbf{C}^8$

- on Q

  normal bundle

  $0 \to T_Q \to T_{\mathsf{P}^7} \to \mathcal{O}(4) \to 0$

  $0 \to \Lambda^4 T_Q(-4) \to \Lambda^4 T_{\mathsf{P}^7}(-4) \to \Lambda^3 T_Q \to 0$

- Coble quartic $Q \subset \mathsf{P}^7$

- $0 \to \mathcal{O}_{\mathsf{P}^7}(-1) \to \mathbf{C}^8 \to T_{\mathsf{P}^7}(-1) \to 0$

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- on Q

  $0 \to T_Q \to T_{\mathsf{P}^7} \to \mathcal{O}(4) \to 0$

  $0 \to \Lambda^4 T_Q(-4) \to \Lambda^4 T_{\mathsf{P}^7}(-4) \to \Lambda^3 T_Q \to 0$

  normal bundle

- $\Rightarrow$ 7-dimensional Heisenberg-invariant subspace of $H^0(\mathcal{N}, \Lambda^3 T)$

- $0 \to \Lambda^4 T_Q(-4) \to \Lambda^4 T_{\mathbb{P}^7}(-4) \to \Lambda^3 T_Q \to 0$

  $0 \to \Lambda^2 T_Q^* \to \Lambda^4 T_{\mathbb{P}^7}(-4) \to \Lambda^3 T_Q \to 0$

- $0 \to H^0(Q, \Lambda^4 T_{\mathbb{P}^7}(-4)) \to H^0(Q, \Lambda^3 T) \to H^1(Q, \Lambda^2 T_Q^*) \to 0$

- $\dim H^1(Q, \Lambda^2 T_Q^*) = g = 3$

- $\dim H^1(\mathcal{N}, \Lambda^2 T^*) = g = 3$

- $h \in H^0(C, K)$

  $\mathcal{O}(\mathsf{End}_0\, E) \xrightarrow{h} \mathcal{O}(\mathsf{End}_0\, E \otimes K)$

  $H^0(C, \mathsf{End}_0\, E \otimes K) \to \mathbb{H}^1 \to H^1(C, \mathsf{End}_0\, E)$

- extension $T^* \to V \to T$ (Courant algebroid)

  class $\in H^1(\mathcal{N}, \Lambda^2 T^*)$ – Heisenberg-invariant

$7 + 3 = 10$ dimensional Heisenberg-invariant sections

$\dim H^1(C, K^{-2}) = 5g - 5 = 10$

injectivity $\Rightarrow$ they all Schouten-commute

From our presentation of $\mathcal{G}$ as defined by linear trinomials and cubic and quartic binomials in $\mathbf{P}^{134}$, we expect that its tropicalization behaves well, especially since $\mathcal{G}$ has a parametrization in terms of a linear map ℓ and a monomial map m:

$$\mathbf{P}^6 \xrightarrow{\ell} \mathbf{P}^{62} \xrightarrow{m} \mathbf{P}^{134}.$$

This approach requires calculating the "Bergman fan" of the (matroid of) the root system E_7, which is currently out of our reach.

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This approach requires calculating the "Bergman fan" of the (matroid of) the root system E_7, which is currently out of our reach.

Happy Birthday, Oscar!