Hitchin map on very stable and even very stable upward flows

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Conference Bernoulli Center, EPFL 18 - 22 January 2016

- Retreat on Higgs bundles, real groups, Langlands duality and mirror symmetry - organized by Oscar García-Prada



Motivation from mirror symmetry

- $\mathcal{M}_{\mathrm{G}}$ moduli of G-Higgs bundles on curve C; G cplx reductive
- classical limit (Donagi–Pantev 2012) of mirror symmetry: $\mathcal{S} : D^b(\mathcal{M}_{\mathrm{G}}) \sim D^b(\mathcal{M}_{\mathrm{G}^\vee})$
- (Kapustin–Witten 2007) complex Lagrangians in $\mathcal{M}_{\mathrm{G}}$ mirror hyperholomorphic vector bundles on $\mathcal{M}_{\mathrm{G}^\vee}$
- examples for Lagrangians:
  1. $\mathcal{M}_{\mathrm{G}_\mathbb{R}} \subset \mathcal{M}_{\mathrm{G}}$ for $\mathrm{G}_\mathbb{R}$ real form of G
  2. Lagrangian upward flows $W^+_{\mathcal{E}} \subset \mathcal{M}_{\mathrm{G}}$
- for (1) (Baraglia–Schaposnik 2016) conjectures support of $\mathcal{S}(\mathcal{O}_{\mathcal{M}_{\mathrm{G}_\mathbb{R}}})$ is in $\mathcal{M}_{\mathrm{G}^\vee_\mathbb{R}} \subset \mathcal{M}_{\mathrm{G}^\vee}$ where complex reductive subgroup $\mathrm{G}^\vee_\mathbb{R} \subset \mathrm{G}^\vee$ *Nadler group* (Nadler 2005)
- for (2) (Kapustin–Witten 2007) dominant $\mu \in \Lambda^+(\mathrm{G}^\vee)$, $c \in C$ *Hecke operators*: $\overline{\mathcal{H}^\mu_c} : D^b(\mathcal{M}_{\mathrm{G}}) \to D^b(\mathcal{M}_{\mathrm{G}})$ and *Wilson operators*: $\mathcal{W}^\mu_c : D^b(\mathcal{M}_{\mathrm{G}^\vee}) \xrightarrow{\otimes \rho_\mu(\mathbb{E}^\vee_c)} D^b(\mathcal{M}_{\mathrm{G}^\vee})$ intertwine: $\overline{\mathcal{H}^\mu_c} \circ \mathcal{S} = \mathcal{S} \circ \mathcal{W}^\mu_c$
- the Hecke transform of the Hitchin section $\mathcal{H}^\mu_c(W^+_0)$ is a Lagrangian upward flow

Bialynicki-Birula decomposition

- X complex, semi-projective, i.e. $\mathbb{C}^\times \circlearrowright X$ such that:
  1. action is linear $\mathbb{C}^\times \circlearrowright X \subset \mathbb{P}^N \circlearrowright \mathbb{C}^\times$
  2. $X^{\mathbb{C}^\times}$ projective
  3. $\lim_{\lambda\to 0} \lambda x$ exists for every $x \in X$
- e.g.: cotangent bundles, Higgs moduli spaces, quiver varieties
- for every $\alpha \in X^{\mathbb{C}^\times}$ denote
  $W_\alpha^+ := \{x \in X | \lim_{\lambda\to 0} \lambda x = \alpha\}$ *upward flow* from α
  $W_\alpha^- := \{x \in X | \lim_{\lambda\to \infty} \lambda x = \alpha\}$ *downward flow* from α
- (Bialynicki-Birula 1973): for $\alpha \in X^{s\mathbb{C}^\times}$ W_α^+, W_α^- locally closed and $W_\alpha^+ \cong T_\alpha^+ X$ and $W_\alpha^- \cong T_\alpha^- X$
- $X = \coprod_\alpha W_\alpha^+$ *BB-partition*
- (X, ω) symplectic s.t. $\lambda^*(\omega) = \lambda\omega \rightsquigarrow T_\alpha^+, T_\alpha^{\leq 0}$, W_α^+, $\coprod_\alpha W_\alpha^-$ Lagrangian
- $C := \coprod_\alpha W_\alpha^-$ *core of* X; $C \sim X$
- $\alpha \in X^{s\mathbb{C}^\times}$, W_α^+ is *very stable* $\Leftrightarrow$ $W_\alpha^+ \subset X$ is closed $\Leftrightarrow$ $W_\alpha^+ \cap C = \{\alpha\}$

Lagrangian upward flows in $\mathcal{M}$

- C smooth, projective, complex curve
- (E, Φ) GL_n-Higgs bundle: E rank n; $\Phi \in \mathrm{End}(E) \otimes K_C$
- $\mathcal{M} := \mathcal{M}^d_{\mathrm{GL}_n}$ moduli of ss degree d GL_n-Higgs bundles
- $\mathbb{C}^\times \circlearrowright \mathcal{M}$ by $(E, \Phi) \mapsto (E, \lambda\Phi)$; semi-projective
- $$\begin{array}{rccc} h: & \mathcal{M} & \to & \mathcal{A} := H^0(K_C) \times H^0(K_C^2) \times \cdots \times H^0(K_C^n) \\ & (E, \Phi) & \mapsto & \det(x - \Phi) \end{array}$$
- *Hitchin map* h is proper, $\mathbb{C}^\times$ equivariant with $\mathbb{C}^\times \circlearrowright \mathcal{A}$ with positive weights $\Rightarrow \mathcal{M}$ semi-projective
- (Hitchin 1987) $\rightsquigarrow$ h is integrable system $\Rightarrow$ (Laumon 1987) $h^{-1}(0)$ *nilpotent cone* is Lagrangian $\cong C$ the core of $\mathcal{M}$
- $\mathcal{M} = \coprod_{\mathcal{E} \in \mathcal{M}^{\mathbb{C}^\times}} W^+_{\mathcal{E}}$; $\mathcal{E} \in \mathcal{M}^{\mathbb{C}^\times}$ *very stable* $\Leftrightarrow$ $W^+_{\mathcal{E}}$ closed $\Leftrightarrow$ $W^+_{\mathcal{E}} \cap h^{-1}(0) = \{\mathcal{E}\} \Leftrightarrow h|_{W^+_{\mathcal{E}}} : W^+_{\mathcal{E}} \to \mathcal{A}$ proper
- very stable Higgs bundles $(E, 0)$ exist by (Laumon 1987)
- Motivating Problem: find coordinates s.t.

$$h_{\mathcal{E}} := h|_{W^+_{\mathcal{E}}} : W^+_{\mathcal{E}} \to \mathcal{A}$$

becomes explicit!

Examples of very stable Higgs bundles

- $E_0 := \mathcal{O}_C \oplus K_C^{-1} \cdots \oplus K_C^{1-n}$
- $a = (a_1, \ldots, a_n) \in \mathcal{A} = H^0(K_C) \times \cdots \times H^0(K_C^n)$
- $\Phi_a := \begin{pmatrix} 0 & 0 & \ldots & 0 & a_n \\ 1 & 0 & \ldots & 0 & a_{n-1} \\ \vdots & & & \vdots & \\ 0 & \ldots & 1 & 0 & a_2 \\ 0 & \ldots & & 1 & a_1 \end{pmatrix} : E_0 \to E_0 K_C$ companion matrix
- $\mathcal{E}_0 := (E_0, \Phi_0) \in \mathcal{M}^{\mathbb{C}^\times}$ *canonical uniformising Higgs bundle*
- upward flow $W_0^+ = \{(E_0, \Phi_a)\}_a$ *Hitchin section* $\Rightarrow$ very stable
- $\delta := (\delta_0, \delta_1, \ldots, \delta_{n-1})$, divisors on C; for $i > 0$ $\delta_i \geq 0$.
  $E_\delta := \mathcal{O}_C(\delta_0) \oplus K_C^{-1}(\delta_0 + \delta_1) \oplus \ldots \oplus K_C^{1-n}(\delta_0 + \delta_1 + \cdots + \delta_{n-1})$,
  $s_{\delta_i} \in H^0(\mathcal{O}_C(\delta_i))$ defining, $\Phi_\delta := \begin{pmatrix} 0 & \ldots 0 & \ldots 0 \\ s_{\delta_1} & \ldots 0 & \ldots 0 \\ \vdots & & \vdots \\ 0 & \ldots s_{\delta_i} & \ldots 0 \\ \vdots & & \vdots \\ 0 & \ldots s_{\delta_{n-1}} & 0 \end{pmatrix} : E_\delta \to E_\delta K_C$
- $c \in C$, $0 < k < n$ and $(0, \ldots 0, \delta_k = c, 0 \ldots) \rightsquigarrow \mathcal{E}_k = (E_k, \Phi_k)$

Theorem (Hausel–Hitchin 2022)

$\mathcal{E}_\delta := (E_\delta, \Phi_\delta) \in \mathcal{M}^s \Rightarrow$ *very stable* $\Leftrightarrow \delta_1 + \cdots + \delta_{n-1}$ *is reduced.*

- e.g. $\mathcal{E}_k$ very stable, in particular $h_k := h|_{W_k^+} : W_k^+ \to \mathcal{A}$ proper

Hecke transformations

Theorem (Hausel–Hitchin 2022)

$\mathcal{E}_\delta := (E_\delta, \Phi_\delta) \in \mathcal{M}^s \Rightarrow$ *very stable* $\Leftrightarrow \delta_1 + \cdots + \delta_{n-1}$ *is reduced.*

- (E, Φ) Higgs bundle $c \in C$, $V \in \mathrm{Gr}(k, E_c)$ s.t. $\Phi(V) \subset VK_c$

$$\begin{array}{ccccccc} 0 \to & E' & \to & E & \to & E_c/V & \to 0 \\ & \Phi' \downarrow & & \Phi \downarrow & & \overline{\Phi}_c \downarrow & \\ 0 \to & E'K & \to & EK & \to & E_c/VK & \to 0 \end{array}$$

  define Hecke transform $\mathcal{H}_V^{\omega_k} := (E', \Phi')$
- observation 1: $\mathcal{E}_\delta$ and W_δ^+ can be obtained by successive Hecke transformations from $\mathcal{E}_0$ and W_0^+
- if $\boldsymbol{\mu} : C \to \Lambda^+(\mathrm{GL}_n) = \{\sum_{i=1}^n a_i\omega_i | a \in \mathbb{Z}_{\geq 0}^{n-1} \times \mathbb{Z}\}$ finite support and $\mathcal{H}^{\boldsymbol{\mu}}$ denotes the matching Hecke correspondence $\Rightarrow$ $\mathcal{H}^{\boldsymbol{\mu}}(W_0^+) = W_{\delta_{\boldsymbol{\mu}}}^+$ where $\delta_{\boldsymbol{\mu}} = (\sum_{c \in C} \langle \boldsymbol{\mu}(c), \omega_{n-i}^\vee \rangle c)_{i=0,\ldots,n-1}$
- observation 2: $\mathcal{E}_{\delta'} \in \overline{W}_\delta^+ \Rightarrow \boldsymbol{\mu}_{\delta'}(c) \leq \boldsymbol{\mu}_\delta(c)\ \forall c \in C$ where $\boldsymbol{\mu}_\delta(c) = \sum_{c \in C} \sum_{i=0}^{n-1} \delta_i(c)\omega_{n-i} \in \Lambda^+$
  $\mu, \lambda \in \Lambda^+$ then $\mu \geq \lambda \Leftrightarrow \lambda - \mu = \sum_{\alpha \in \Delta^+} a_\alpha \alpha$ with $a_\alpha \in \mathbb{Z}_{\geq 0}$
- $\Rightarrow \mathcal{E}_\delta \in \mathcal{M}^s$ very stable $\Leftrightarrow \boldsymbol{\mu}_\delta(c) \in \Lambda^+$ minuscule $\forall c \in C \Rightarrow$ □

Hitchin map as spectrum of equivariant cohomology

- G complex reductive; $E\mathrm{G} \to B\mathrm{G}$ universal principal G-bundle; $H^*_{\mathrm{G}} := H^*(B\mathrm{G}; \mathbb{C}) \cong \mathbb{C}[\mathfrak{t}]^{\mathrm{W}}$
- $\mathrm{G} \curvearrowright X$ variety; $X_{\mathrm{G}} := X \times E\mathrm{G}/\mathrm{G}$ *Borel quotient*; $H^*_{\mathrm{G}}(X; \mathbb{C}) := H^*(X_{\mathrm{G}}; \mathbb{C})$ *equivariant cohomology* H^*_{G}-algebra
- *equivariantly formal* $\Leftrightarrow H^*_{\mathrm{G}}(X; \mathbb{C})_0 \cong H^*(X; \mathbb{C}) \Leftrightarrow H^*_{\mathrm{G}}(X)$ is free over $H^*_{\mathrm{G}} \Rightarrow \mathrm{Spec}(H^{2*}_{\mathrm{G}}(X; \mathbb{C})) \to \mathrm{Spec}(H^{2*}_{\mathrm{G}}) \cong \mathfrak{t}/\!\!/\mathrm{W}$ is proper

Theorem (Hausel, 2022)

The Hitchin map on the minuscule upward flow W_k^+ is modelled on spectrum of GL_n-equivariant cohomology of the Grassmannian

$$\begin{array}{rcc} W_k^+ & \twoheadrightarrow & \mathrm{Spec}(H^{2*}_{\mathrm{GL}_n}(\mathrm{Gr}(\mathrm{k},\mathrm{n}), \mathbb{C})) \\ h_k \downarrow & \lrcorner & \downarrow \\ \mathcal{A} & \twoheadrightarrow & \mathrm{Spec}(H^{2*}_{\mathrm{GL}_n}) \end{array}$$

- proof by fixed point scheme in $\mathrm{Gr}(\mathrm{k},\mathrm{n}) \cong \mathrm{Gr}^{\omega_{\mathrm{k}}}$ *affine Schubert*
- $\rightsquigarrow$ *multiplicity algebra* $\mathbb{C}[h_k^{-1}(0)] \cong H^{2*}(\mathrm{Gr}(\mathrm{k},\mathrm{n}); \mathbb{C})$ matches equivariant multiplicity $m_k(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$ of (Hausel–Hitchin)

Even very stable upward flows

- $\theta(E,\Phi) := (E,-\Phi)$ involution $\mathcal{M}^\theta = \bigcup_p \mathcal{M}_{U(p,n-p)}$ Lagrangian from (García-Prada–Ramanan, 2019)
  and $h^\theta := h|_{\mathcal{M}^\theta} : \mathcal{M}^\theta \to \mathcal{A}^\theta = H^0(K_C^2) \times \dots H^0(K_C^{2\lfloor \frac{n}{2} \rfloor})$
- Problem: study upward flows in $\mathcal{M}^\theta$ and h^θ restricted on them!
- $\mathcal{E} \in \mathcal{M}^{s\mathbb{C}^\times}$, recall $W_\mathcal{E}^+ \cong_{\mathbb{C}^\times} T_\mathcal{E}^+$ define $W_\mathcal{E}^{2+} := (W_\mathcal{E}^+)^\theta \cong_{\mathbb{C}^\times} T_\mathcal{E}^{2+}$
- $\mathcal{E} \in \mathcal{M}^{s\mathbb{C}^\times}$ is *even very stable* $\Leftrightarrow$ $W_\mathcal{E}^{2+} \subset \mathcal{M}^\theta$ is closed $\Leftrightarrow$ $W_\mathcal{E}^{2+} \cap h^{-1}(0) = \{\mathcal{E}\}$
- define even partial ordering on $\Lambda^+ \ni \lambda, \mu$:
  $\alpha \leq_2 \beta \Leftrightarrow \lambda - \mu = \sum_{\alpha \in \Delta^{2+}} a_\alpha \alpha$
  with $a_\alpha \in \mathbb{Z}_{\geq 0}$ and Δ^{2+} positive roots of even height

Theorem (González-González–Hausel, 2022)

Let $\boldsymbol{\mu} : C \to \Lambda^+$ then $\mathcal{E}_{\delta_{\boldsymbol{\mu}}} \in \mathcal{M}^{s\mathbb{C}^\times}$ is even very stable $\Leftrightarrow \forall c \in C$ the dominant weight $\boldsymbol{\mu}(c)$ is even minuscule (= minimal wrt $\leq_2$).

- for GL_2 : $\Delta^{2+} = \emptyset \rightsquigarrow$ all $\mathcal{E}_\delta \in \mathcal{M}^{s\mathbb{C}^\times}$ even very stable
- for GL_4 if $\boldsymbol{\mu}$ trivial except $\boldsymbol{\mu}(c) = 2\omega_1 + 2\omega_3 \rightsquigarrow$ even minuscule but not minuscule $\rightsquigarrow \mathcal{E}_{\delta_{\boldsymbol{\mu}}}$ even very stable but not very stable

Hitchin map on even very stable upward flows

- for simplicity work with GL_{2n} and an even number $0 < 2k < 2n$
- Hitchin map on minuscule upward flow $h_{2k} : W^+_{2k} \to \mathcal{A}$
  modelled by $\mathrm{Spec}(H^{2*}_{\mathrm{GL}_{2n}} \to H^{2*}_{\mathrm{GL}_{2n}}(\mathrm{Gr}(2k,2n),\mathbb{C}))$
  θ acts as $-1 \in \mathbb{C}^*$ via the grading on $H^{2*}_{\mathrm{GL}_n}(\mathrm{Gr}(2k,2n),\mathbb{C})$
  $\rightsquigarrow h^\theta_k : W^{2+}_{2k} \to \mathcal{A}^\theta$ is modelled on
  $$\mathrm{Spec}\left(\frac{H^{2*}_{\mathrm{GL}_{2n}}}{\langle H^{2(2*+1)}_{\mathrm{GL}_{2n}}\rangle} \to \frac{H^{2*}_{\mathrm{GL}_{2n}}(\mathrm{Gr}(2k,2n),\mathbb{C})}{\langle H^{2(2*+1)}_{\mathrm{GL}_{2n}}(\mathrm{Gr}(2k,2n),\mathbb{C})\rangle}\right) \cong$$
  $\mathrm{Spec}(H^{2*}_{\mathrm{Sp}_n} \to H^{2*}_{\mathrm{Sp}_n}(\mathrm{Sp}_n/(\mathrm{Sp}_k \times \mathrm{Sp}_{n-k}),\mathbb{C}))$
  equivariant cohomology of *quaternionic Grassmannian*
- θ-fixing for SO_{4n+2}-minuscule even quadric
  $H^{2*}_{\mathrm{SO}_{4n+2}}(\mathrm{SO}_{4n+2}/\mathrm{SO}_2 x \mathrm{SO}_{4n},\mathbb{C})$ is $H^{2*}_{\mathrm{SO}_{4n+1}}(\mathrm{SO}_{4n+1}/\mathrm{SO}_{4n},\mathbb{C})$
  equivariant cohomology of the *sphere* S^{4n}
- θ-fixing for the E_6-minuscule complex Cayley plane
  $H^{2*}_{E_6}(E_6/(\mathrm{Spin}_{10} \times \mathrm{GL}_1),\mathbb{C})$ is $H^{2*}_{F_4}(F_4/\mathrm{Spin}_9,\mathbb{C})$
  equivariant cohomology of *real Cayley plane*
- Note Nadler groups: $\mathrm{Sp}_n \cong U(n,n)^\vee \subset \mathrm{GL}_{2n}$
  $\mathrm{SO}_{4n+1} \cong \mathrm{SO}(2n,2n+2)^\vee \subset \mathrm{SO}_{4n+2}$ $F_4 \cong E^\vee_{6(2)} \subset E_6$

Problems

- What is the relation of these symmetric spaces of the Nadler group with the (Baraglia–Schaposhnik 2016) conjecture for the mirror of $\mathcal{M}_{\mathrm{G}_{\mathbb{R}}}$?
- Describe the Hitchin map on the other even very stable type $(1, ..., 1)$ upward flows.
- Define multiplicity for even upward flows, compute it for even very stable ones, more generally, prove that $m((\overline{\mathcal{H}^{\mu}}(W_0^+))^{\theta}) = D_{\mu}(-1)$, compare it to the expected $m((\overline{\mathcal{H}^{\mu}}(W_0^+))) = D_{\mu}(1) = \dim(V_{\mu})$, where $D_{\mu}(q) = \prod_{\alpha\in\Delta^+} \frac{1-q^{\langle\mu+\rho,\alpha^\vee\rangle}}{1-q^{\langle\mu,\alpha^\vee\rangle}}$ *Dynkin polynomial* of irrep V_{μ}.
- Prove $m(\overline{\mathcal{H}^{\mu}}(W_0^+))(q) = D_{\mu}(q)$, define equivariant multiplicity for even upward flows, and compute $m((\overline{\mathcal{H}^{\mu}}(W_0^+))^{\theta})(q)$.
- We have conjectural G-representation theory meaning of the algebra $\mathbb{C}[\overline{\mathcal{H}^{\mu}}(W_0^+)]$. What is the $\mathrm{G}_{\mathbb{R}}$-representation theory meaning of $\mathbb{C}[\overline{\mathcal{H}^{\mu}}(W_0^+)^{\theta}]$?
- Study analoguous questions for involutions on $\mathcal{M}_{\mathrm{G}}$ arising from other (quasi-split) real forms.