Components of G-Higgs bundle moduli and the Cayley correspondence

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Partially supported by national funds by FCT (Portugal) through the projects UIDB/00144/2020 and EXPL/MAT-PUR/1162/2021

Conference in honour of Oscar García-Prada on the occasion of his 60th birthday — ICMAT, Madrid, ~~12~~ 16/09/2022



Durália, Porto, 2/6/2017

Recital de Canto





15 de abril de 2018 17:30
Biblioteca do Fundo Antigo
Reitoria da Universidade do Porto

Oscar García-Prada (contra-tenor)
David Mason (piano).

Obras de Byrd, Caccini, Campion, Cavalli, Dowland, Händel, Lambert, Marín, Monteverdi, Purcell, Quilter, Vaughan Williams

Acesso livre, condicionado ao número de lugares disponíveis

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G-Higgs bundles

- G - real reductive Lie group (connected)

  $\cup$

  H - max! compact
- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ - Cartan decomposition; $H \curvearrowright \mathfrak{m}$ via Ad

Def **G-Higgs bundle** (E, φ) on Riemann surface X:

- $E \to X$ principal $H^{\mathbb{C}}$-bundle
- $\varphi \in H^0(E[\mathfrak{m}^{\mathbb{C}}] \otimes K)$

$E[\mathfrak{m}^{\mathbb{C}}] = E \times_{H^{\mathbb{C}}} \mathfrak{m}^{\mathbb{C}}$

$K = \Omega^1 X$

Invariant: $c(E) \in H^2(X, \pi_1 H)$

Moduli space

$$\mathcal{M}(X,G) := \{(E,\varphi) \mid \text{polystable}\}/\simeq$$

A few reasons why this is interesting:

- Hyper-Kähler • Integrable system; Hitchin map
- Mirror symmetry • Higher Teichmüller spaces
- Non-abelian Hodge Theorem:

$$\mathcal{M}(X,G) \qquad \bar\partial_A \varphi = 0$$

$$\cong \{(A,\varphi) \mid F(A) + [\varphi, \varphi^{*_h}] = 0\}/\text{gauge}$$

$$\cong \mathrm{Hom}(\pi_1 X, G)/G \cong H^1(X, \underline{G}) \leftarrow$$

non-abelian 1st cohomology = flat bundles

$$\varphi$$

$$\rho : \pi_1 X \longrightarrow G$$

What is $\pi_0 \mathcal{M}(X, G)$?

Theorem [Li '93], [Garcia-Prado – Oliveira '17] For G complex,

$\mathcal{M}_c(X, G)$ is connected.

ii $\{(E, \varphi) \mid c(E) = c \in H^2(X, \pi_1 H)\}$

Example ($G = SL(2, \mathbb{R})$)

$$(E = L \oplus L^{-1}, \varphi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}), \quad c(E) = \deg(L) \in \mathbb{Z}$$

- Milnor-Wood: $|c(E)| \leq g - 1$
- $\mathcal{M}_c$ is connected, $|c| < g - 1$
- $\mathcal{M}_{g-1}$ has $2g-2$ components, identified in $\mathcal{M}(X, PSL(2, \mathbb{R}))$ with Teichmüller space. [Hitchin, Goldman]

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Example ($G = SL(2, \mathbb{R})$) $PSL(2, \mathbb{R}) = PSU(1,1) = SO_0(1,2)$!

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Hitchin components (G split real)

- $\Psi : \bigoplus_{j=1}^{\mathrm{rk}(\mathfrak{g})} H^0(X, K^{m_j+1}) \hookrightarrow \mathcal{M}(X, G)$ [Hitchin '92]

Example: ($G = SL(2,\mathbb{R})$)

$$H^0(X, K^2) \hookrightarrow \mathcal{M}(X, SL(2,\mathbb{R}))$$
$$q \longmapsto \left(K^{1/2} \oplus K^{-1/2}, \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}\right)$$

Remarks (1) Ψ is a section of the Hitchin map

(2) Generalised to Hitchin-Kostant-Rallis section for any real reductive G [Garcia-Prada, Peon-Nieto, Ramanan '17]

Non-compact G of Hermitian type

- $SU(p,q)$, $Sp(2n,\mathbb{R})$, $SO^*(2n)$, $SO_0(2,n)$, E_7^{-25}, E_6^{-14}

Toledo invariant: $\boxed{\tau(E,\varphi) \in \mathbb{Z}}$

MW: $|\tau(E,\varphi)| \leq \mathrm{rk}(G/H)(2g-2)$

Def (E,φ) is maximal if $|\tau(E,\varphi)| = \mathrm{rk}(G/H)(2g-2)$

Example $G = SU(p,q)$ $\left(E = V \oplus W,\ \varphi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}\right)$

$$\tau = 2\deg(V)$$

Cayley correspondence (G of tube type, adjoint)

[Bradlow, García-Prada, G., Mundet] classical cases

[Biquard, García-Prada, Rubio '17] general

- $\Psi : \mathcal{M}_{KH}(X, G_{\mathbb{C}}) \xrightarrow{\cong} \mathcal{M}_{max}(X, G)$
- New topological invariants for $\mathcal{M}_{max}(X,G)$
  $\Rightarrow$ "extra" connected components
- The non-tube case reduces to the tube case via rigidity

Example $G = SU(p,p)$ $\left(E = V \oplus W,\ \varphi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}\right)$

maximal: $\gamma : V \xrightarrow{\cong} W^*$, $\Psi = \beta \circ \gamma : V K^{-1/2} \longrightarrow V K^{3/2}$

The non-maximal case

Conjecture For non-compact G of Hermitian type, $\mathcal{M}_c(X, G)$ is connected when c is not maximal

Known cases:

- $U(p,q)$ [Bradlow-García-Prada-G. '03] [BGG-Heinloth '18]
- $Sp(2n, \mathbb{R})$ $n = 1, 2$ [García-Prada - Mundet '04] [G.-Oliveira]

Possible approach: wall-crossing for principal pairs

The case $G = SO(p,q)$

- [M. Aparicio Arroyo PhD '09] - indication of "extra" components!
- $SO(p,p+1)$ - [Collier '20] parametrisation of "extra" components which DO NOT deform to $\mathcal{M}_c(X,H)$
  → common feature of Hitchin and maximal

Theorem [Aparicio-Bradlow-Collier-García Prada-G-Oliveira '19]

(1) There is an open and closed injective map

$$\mathcal{M}_{KP}(X, SO(1,q-p+1)) \times \bigoplus_{j=1}^{p-1} H^0(X, K^{2j}) \hookrightarrow \mathcal{M}(SO(p,q))$$

(2) Determined $\pi_0 \mathcal{M}(X, SO(p,q))$ for $p \neq 2$

General mechanism: Magical nilpotents [BCGGO] arxiv: 2101.09377

- $\mathfrak{g}$ Lie algebra / $\mathbb{C}$ ⚠ Change of notation $\mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}$
- Nilpotent $e \in \mathfrak{g}$: $ad(e): \mathfrak{g} \to \mathfrak{g}$ nilpotent

Jacobson-Morozov Theorem: e can be completed to

- $\{f, h, e\}$ – sl_2-triple: $[h,e]=2e$, $[h,f]=-2f$, $[e,f]=h$
- $sl_2\mathbb{C} \cong \langle f, h, e \rangle \subseteq \mathfrak{g}$ $\quad \left| \begin{array}{c} \{e \neq 0 \in \mathfrak{g} \text{ nilpotent}\}/G \\ \overset{1-1}{\longleftrightarrow} \{\phi : sl_2\mathbb{C} \hookrightarrow \mathfrak{g}\}/G \end{array} \right.$

Decompositions of $\mathfrak{g}$ (associated to $\{f, h, e\}$):

- $\mathfrak{sl}_2\mathbb{C}$-module: $\mathfrak{g} = \bigoplus_{j=0}^{N} W_j$; $W_j = S^j(\mathbb{C}^2)^{\oplus n_j}$

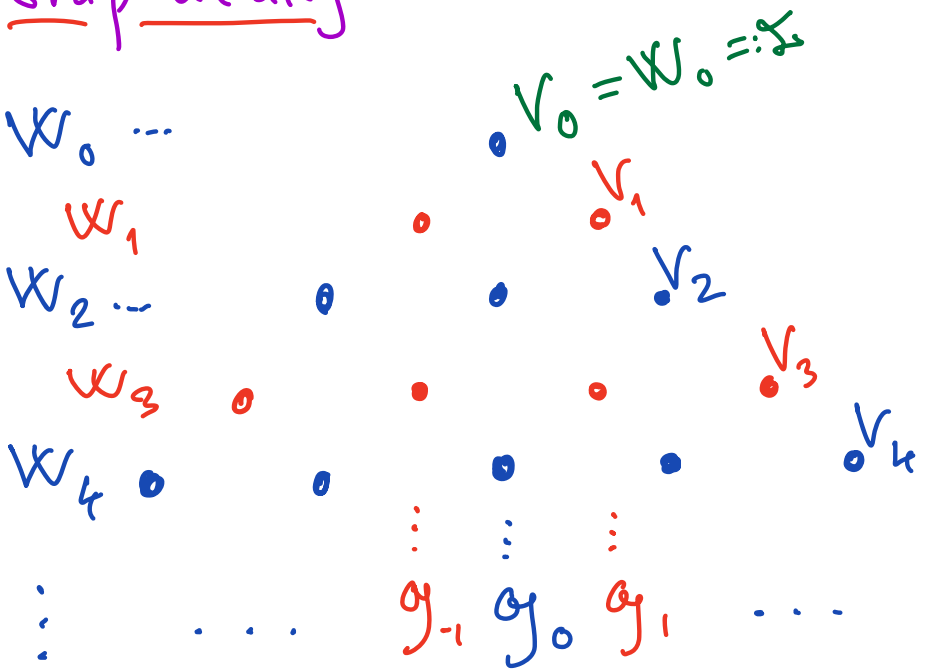
  ($S^j(\mathbb{C}^2)$: fundamental $\mathfrak{sl}_2\mathbb{C}$-rep.)

- ad_h-weight spaces: $\mathfrak{g} = \bigoplus_{j=-\ell}^{\ell} \mathfrak{g}_j$; $[h, x] = j \cdot x$; $x \in \mathfrak{g}_j$

- $V := \ker(\mathrm{ad}_e) = \bigoplus_{j \geq 0} V_j \subseteq \mathfrak{g}$; $V_j = W_j \cap \mathfrak{g}_j$

  (V: centraliser of e; V_j: highest weight space in rep^n W_j.)

Graphically



Facts:

- $\Sigma := V_0 = W_0 = \mathfrak{g}^{<f,h,e>}$
- $x \in \Sigma$

  $\Rightarrow [x, W_j \cap \mathfrak{g}_k] \subseteq W_j \cap \mathfrak{g}_k$
- $ad_f : W_j \cap \mathfrak{g}_k \xrightarrow{\cong} W_j \cap \mathfrak{g}_{k-2}$

  $-j+2 \leq k \leq j$

Define involution $\sigma : \mathfrak{g} \longrightarrow \mathfrak{g}$ by:

$$\sigma(x) = \begin{cases} x, & x \in V_0 \\ (-1)^{k+1} x, & x \in ad_f^k(V_j), \ j > 0 \end{cases}$$

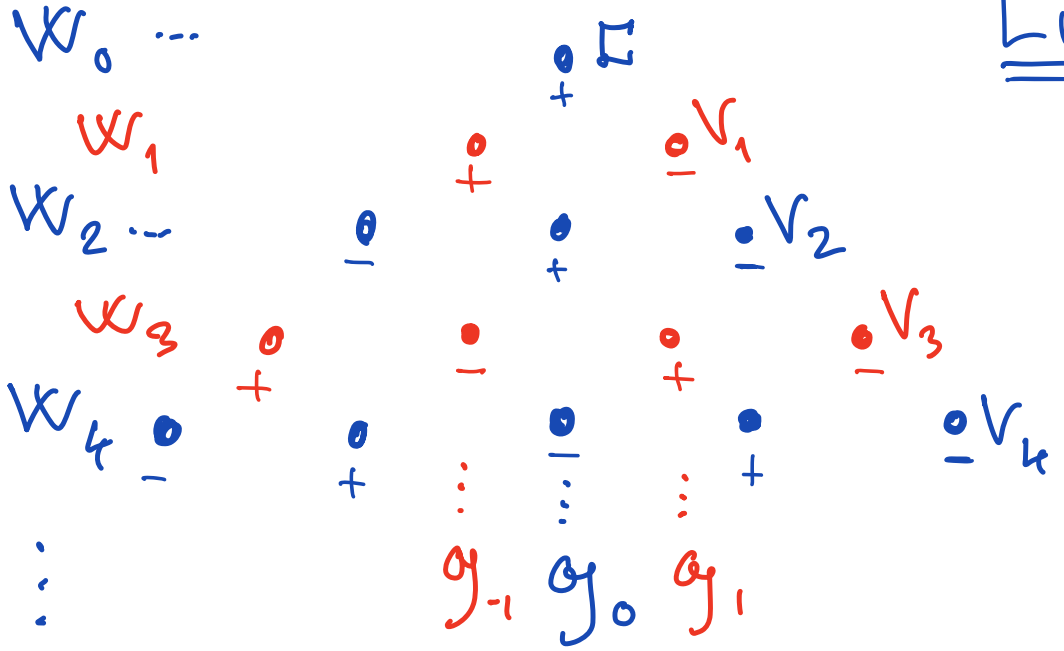
Note $\sigma|_{V_0} = Id$

$\sigma|_{V_j} = -Id$, $j > 0$

Magical nilpotents

Def: e (or $\{f,h,e\}$) is **magical** if $\sigma : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra involution:

$$[\sigma(x), \sigma(y)] = \sigma([x,y])$$



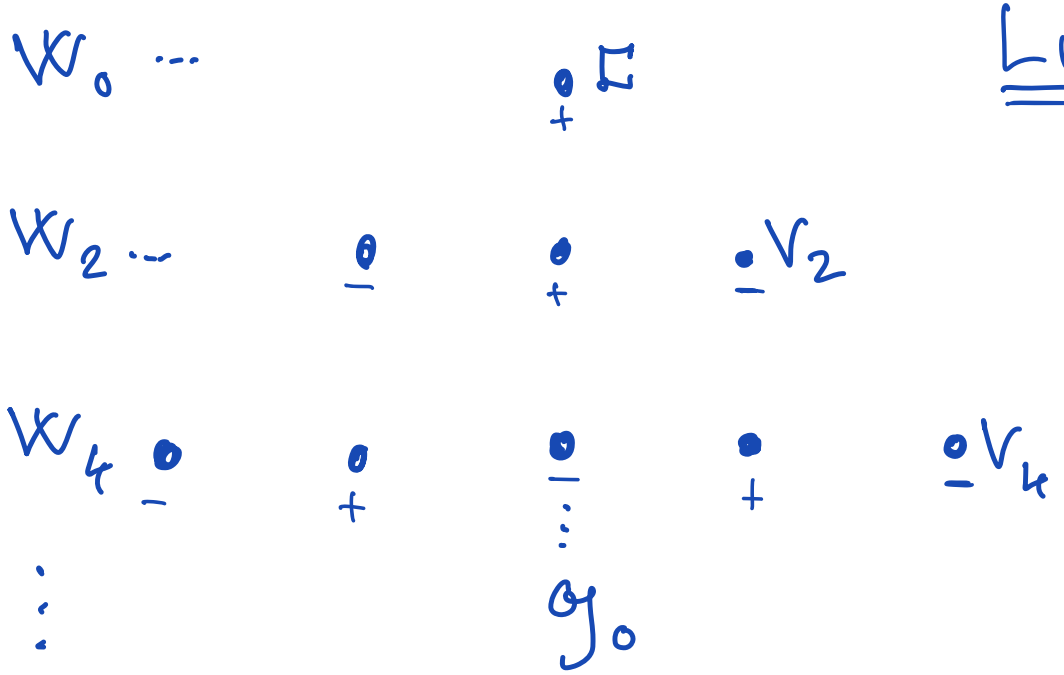
Example: if e is principal, then it is magical [Kostant]

- φ_+ from the HSS Cayley correspondence

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Rmk Magical

$\Rightarrow W_{2k+1} = 0$

"e is even"

Example: if e is principal, then it is magical [Kostant]

- φ_+ from the HSS Cayley correspondence

Magical nilpotents (cont.)

- **Canonical real form:**

  (*) $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ – Cartan decomposition associated to σ

  Def: The **canonical real form** associated to the magical $\mathfrak{sl}_2\mathbb{C}$ triple $\{f, h, e\}$ is $\mathfrak{g}^{\mathbb{R}} \subseteq \mathfrak{g}$ given by (*)

- **Classification Theorem** for magical nilpotents (using classification of nilpotents in real Lie algebras)

Canonical real forms

Theorem [BCGGPO]
The canonical real forms associated to magical nilpotents in simple complex Lie algebras are:

(1) Split real forms

(2) Hermitian of tube type

(3) $so(p,q)$

(4) E_6^2, E_7^{-5}, E_8^{-24}, F_4^4 (quaternionic real forms)

Rmk This coincides with the list of groups admitting a positive structure [Guichard-Wienhard '16]

The sub-algebra $\mathfrak{g}_0 \subseteq \mathfrak{g}$ and the Cayley real form

- $\mathfrak{g}_0 = I \oplus \bigoplus_{j=1}^{N} Z_{2m_j}$ $\quad (Z_{2m_j} := \mathfrak{g}_0 \cap W_{2m_j})$

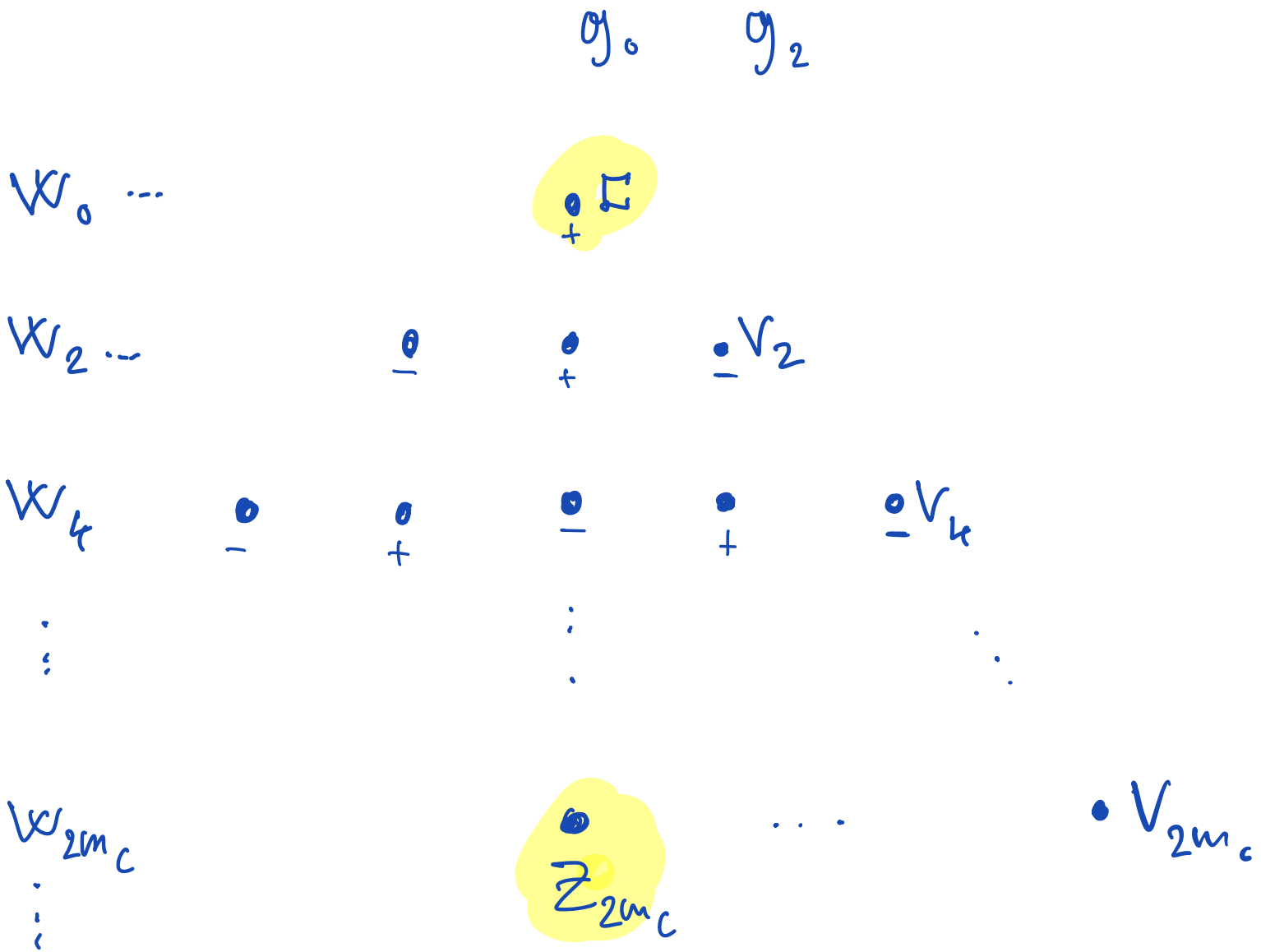
Prop. e magical $\iff$ $[Z_{2m_i}, Z_{2m_j}] \subset I \quad \forall i,j$

Further consequences:

- $[Z_{2m_i}, Z_{2m_j}] = 0 \quad \forall i \neq j$
- There is at most one $c \in \{1,..,N\}$ s.t. $[Z_{2m_c}, Z_{2m_c}] \neq 0$

$(\iff n_{2m_c} > 1)$

Graphically



The Cayley real form

$$\Rightarrow \mathfrak{g}_0 = \mathfrak{k} \oplus \bigoplus_{j=1}^{N} Z_{2m_j} \quad \text{is a Cartan decomposition}$$

$$= \underbrace{\mathfrak{k} \oplus Z_{2m_c}}_{\mathfrak{g}_e \ (s.s.)} \oplus \bigoplus_{j \neq c} \underbrace{Z_{2m_j}}_{\mathbb{C}^{N-1}}$$

Def. The Cayley real form associated to the magical nilpotent e is the associated real form of $\mathfrak{g}_0$.

Rmk: This is split $\Leftrightarrow \mathfrak{k} = 0$

Def. The Cayley-partner is $\mathfrak{g}_e^{\mathbb{R}} \subseteq \mathfrak{g}_e$.

Back to Higgs bundles

- $G^{\mathbb{R}} \subset G$ canonical real form $\leftarrow$ magical $\{f, h, e\}$

Want to define:

$$\Psi : \mathcal{M}_{K^{m_c+1}}(X, G_e^{\mathbb{R}}) \times \bigoplus_{\substack{j=1 \\ j \neq c}}^{N} H^0(K^{m_j+1}) \longrightarrow \mathcal{M}(X, G^{\mathbb{R}})$$

$$(E_C, \psi_{m_c}), \psi_{m_1}, \ldots, \psi_{m_N}) \mapsto (E_T * E_C[H], f + \sum_{j=1}^{N} \varphi_{m_j})$$

where: $T \subseteq S \subseteq G \supseteq C$ (integrating Lie algebras)

$\mathbb{C}^* \quad PSL(2,\mathbb{C}) \quad H$

Construction of bundle

- (E_T, f) uniformising $PSL(2,\mathbb{R})$-Higgs bundle
  - coming from $(K^{\frac{n}{2}} \oplus K^{-\frac{n}{2}}, \begin{pmatrix} 0 & \varphi \\ 1 & 0 \end{pmatrix})$
- $E_C \to X$ principal C-bundle

Define:

$$E_H := E_C \star E_T[H] := (E_C \times_X E_T)[H] \quad (\text{is a } H\text{-bundle})$$

Then:

$$E_H[\mathfrak{g}] \cong \bigoplus_j E_C \star E_T[\mathfrak{g}_j] =: E_H[\mathfrak{g}]_j$$

$$\| 12$$

$$E_H[\mathfrak{h}] \oplus E_H[\mathfrak{m}]$$

Construction of Higgs field

- $E_C \overset{*}{\in} E_T[\langle f \rangle] \subseteq E_H[\mathcal{O}]_{-2}$

$E_T[\langle f \rangle] \cong K^{-1} \Rightarrow f \in H^0(E_C[\mathcal{O}]_{-2} \otimes K)$

$E_T[\langle f \rangle] \otimes K$

$\cap 1$

- $E_C \overset{*}{\in} E_T[v_j] \subseteq E_H[\mathcal{O}]_j$

Isomorphisms:

(*) $ad_f^{m_j} : E_C \overset{*}{\in} E_T[v_{2m_j}] \otimes K \xrightarrow{\cong} E_C[Z_{2m_j}] \otimes K^{m_j+1}$

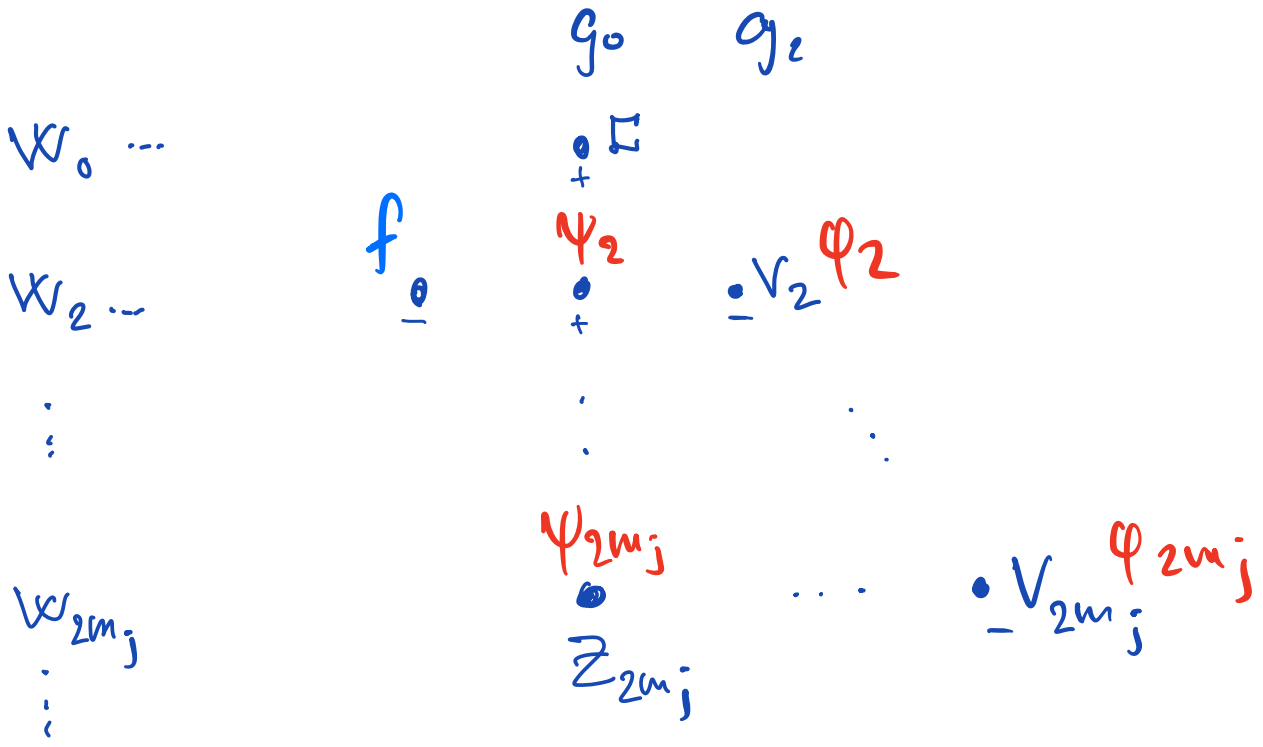
- G_C^R - Higgs bundle: $(E_C, \psi_{m_c} \in H^0(E_C[Z_{2m_c}] \otimes K^{m_c+1})$
- R^t - Higgs bundles: $\psi_{m_j} \in H^0(K^{m_j+t})$

(K^{m_j+1} - twisted!)

Using (*) we can now define:

$$H^0(E_C * E_T[V_{2m_j}] \otimes K) \xrightarrow{\cong} H^0(E_C[Z_{2m_j}] \otimes K^{m_j+n})$$

$$\varphi_{m_j} \longmapsto \psi_{m_j} = ad_f^{m_j}(\varphi_{m_j})$$



Theorem The map

$$\Psi : \mathcal{M}_{K^{m_{c+1}}}(X, G_e^{\mathbb{R}}) \times \bigoplus_{\substack{j=1 \\ j \neq c}}^{N} H^0(K^{m_{j+1}}) \longrightarrow \mathcal{M}(X, G^{\mathbb{R}})$$

is well-defined, injective, and open and closed.

About the proof:

- It relies heavily on the classification theorem
- Ψ well defined (preservation of stability) is delicate
- openness uses local analysis and $H^2 = 0$
- closedness uses properness of the Hitchin map

Geometric properties of surface group representations (II story)

NAHT: $\mathcal{M}(X, G^{\mathbb{R}}) \cong \mathrm{Hom}(\pi_1 X, G^{\mathbb{R}}) /\!\!/ G^{\mathbb{R}}$

- Anosov representations [Labourie '06] ⇒ discrete and faithful
  - Hitchin reps are Anosov
  - Maximal reps are Anosov [Burger-Iozzi-Labourie-Wienhard '06]

- P-positive representations [Guichard-Wienhard '16]
  - classified $G^{\mathbb{R}}$ admitting positive structure (= magical list)

Higher Teichmüller space: component consisting entirely of positive reps

[Also Fock-Goncharov]

Higher Teichmüller spaces

- We show that Cayley components
  - contain positive reps;
  - do not contain reps that factor through a parabolic.
- [GLW '21]: P-positive reps
  - are Anosov.
  - are open in $\mathrm{Hom}(\Gamma, S, G^{\mathbb{R}})/\!/G^{\mathbb{R}}$;
  - are closed in the subset of reps which do not factor through a parabolic.

$\Rightarrow$ Cayley components are higher Teichmüller spaces!

(Also: Beyrer-Pozzetti)

What is $\pi_0 \mathcal{M}(X, G)$?

Conjecture: All "extra" components are Cayley exponents

Known for classical groups, except for non-maximal Hermitian (except $SU(p,q)$).

Counting components of $\mathcal{M}(X,G)$ (Hitchin's method)

Hitchin functional: $f : \mathcal{M} \longrightarrow \mathbb{R}$

$$(E,\varphi) \longmapsto \int_X |\varphi|^2$$

- uses harmonic metric on (E,φ)
- perfect Bott-Morse function if $\mathcal{M}$ is smooth
  - moment map of S^1-action $(E,\varphi) \mapsto (E, e^{i\theta}\varphi)$

$\rightsquigarrow$ critical points of f are fixed points

- From holomorphic point of view:

$\pi_0 \mathcal{M}(X,G)$ is determined by loci of minimal fixed points of $\mathbb{C}^*$-action. $\mathbb{C}^* \times \mathcal{M} \longrightarrow \mathcal{M}$, $(t,(E,\varphi)) \longmapsto (E, t\varphi)$

Fixed points

$\{\varphi = 0\}$
$\cup$
(1) $\varphi = 0 \iff \mathcal{M}(X,H) \subset \mathcal{M}(X,G)$

(2) $0 \neq \varphi$ is nilpotent and have reduction $E_0 \subset E$ to $H_0^c \subset H^c$ with

$$\mathfrak{g} = \bigoplus \mathfrak{g}_j \qquad \text{grading by } h \in \mathfrak{h}_0;$$
$$= \bigoplus \mathfrak{h}_j \oplus \mathfrak{m}_j \qquad \varphi \in H^0(E_0[\mathfrak{m}_{-1}])$$

Thm. [BGG '03] A fixed point is minimal

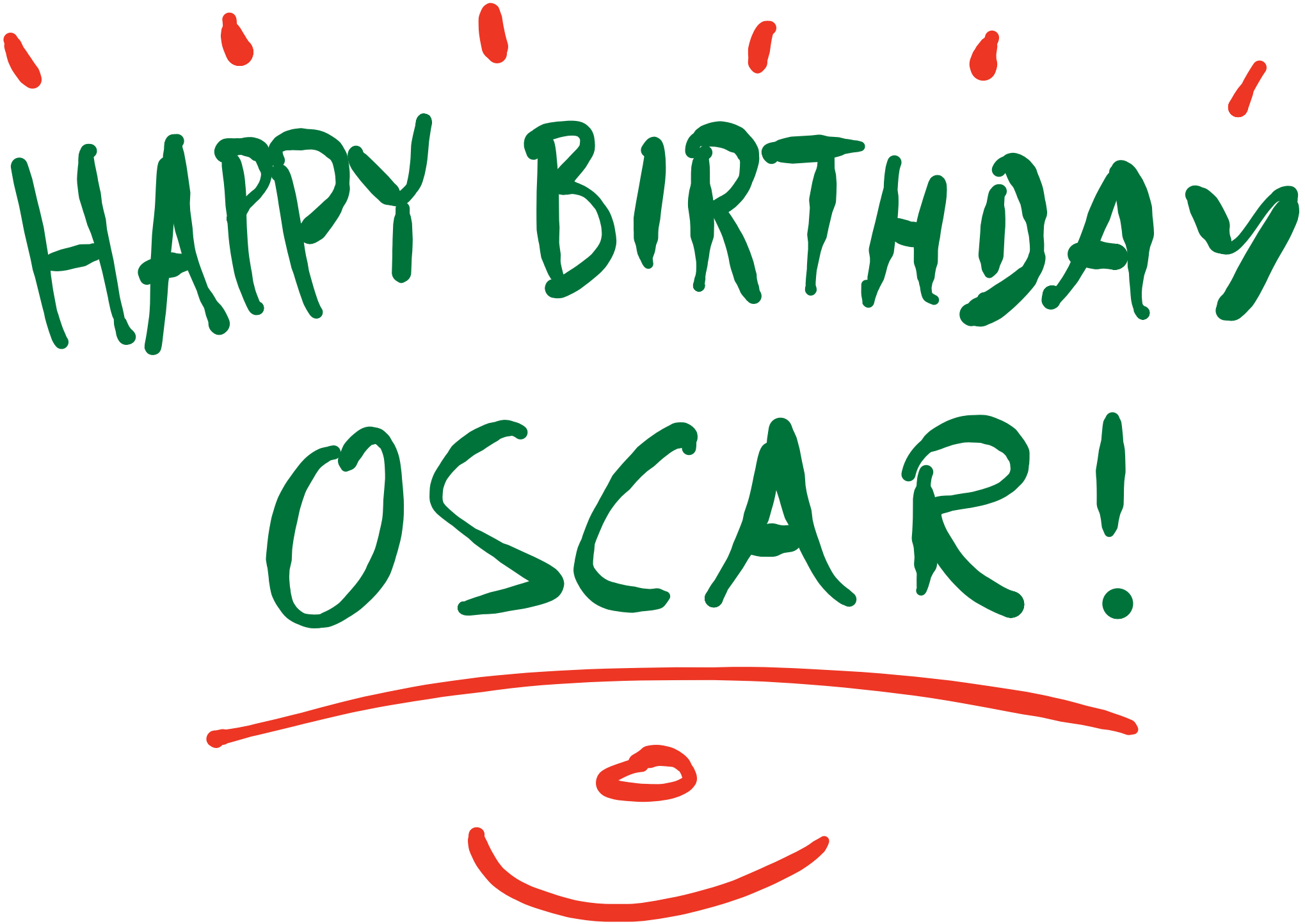
$\iff ad_\varphi : E_0[\mathfrak{h}_j] \xrightarrow{\cong} E_0[\mathfrak{m}_{j-1}] \otimes K \quad \forall j < 0$

One find result

Theorem Let $\mathfrak{g}$ be a complex simple Lie algebra with a Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and let $\{f, h, e\}$ be an $\mathfrak{sl}_2$-triple s.t.

$$\mathrm{ad}_f : \mathfrak{h}_j \xrightarrow{\cong} \mathfrak{m}_{j-2} \qquad \forall\, j < 0$$

(where the grading is given by h).
Then $\{f, h, e\}$ is maximal with canonical real form given by $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.


HAPPY BIRTHDAY
OSCAR!