Futaki invariants and harmonic metrics for the Hull-Strominger system

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ICMAT, 14th September 2022

Joint work with Raúl González Molina (UAM-ICMAT), *to appear*.

A family picture



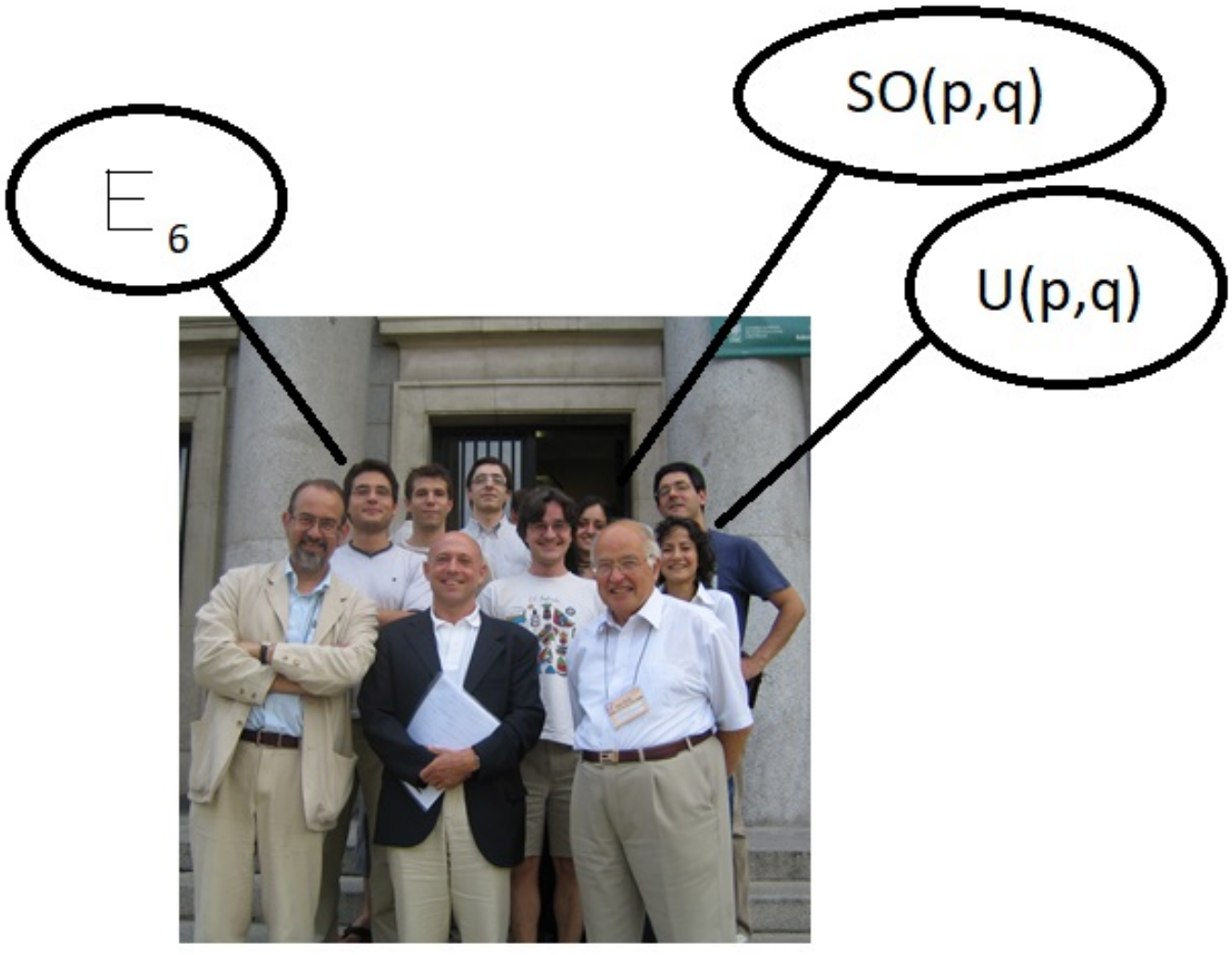
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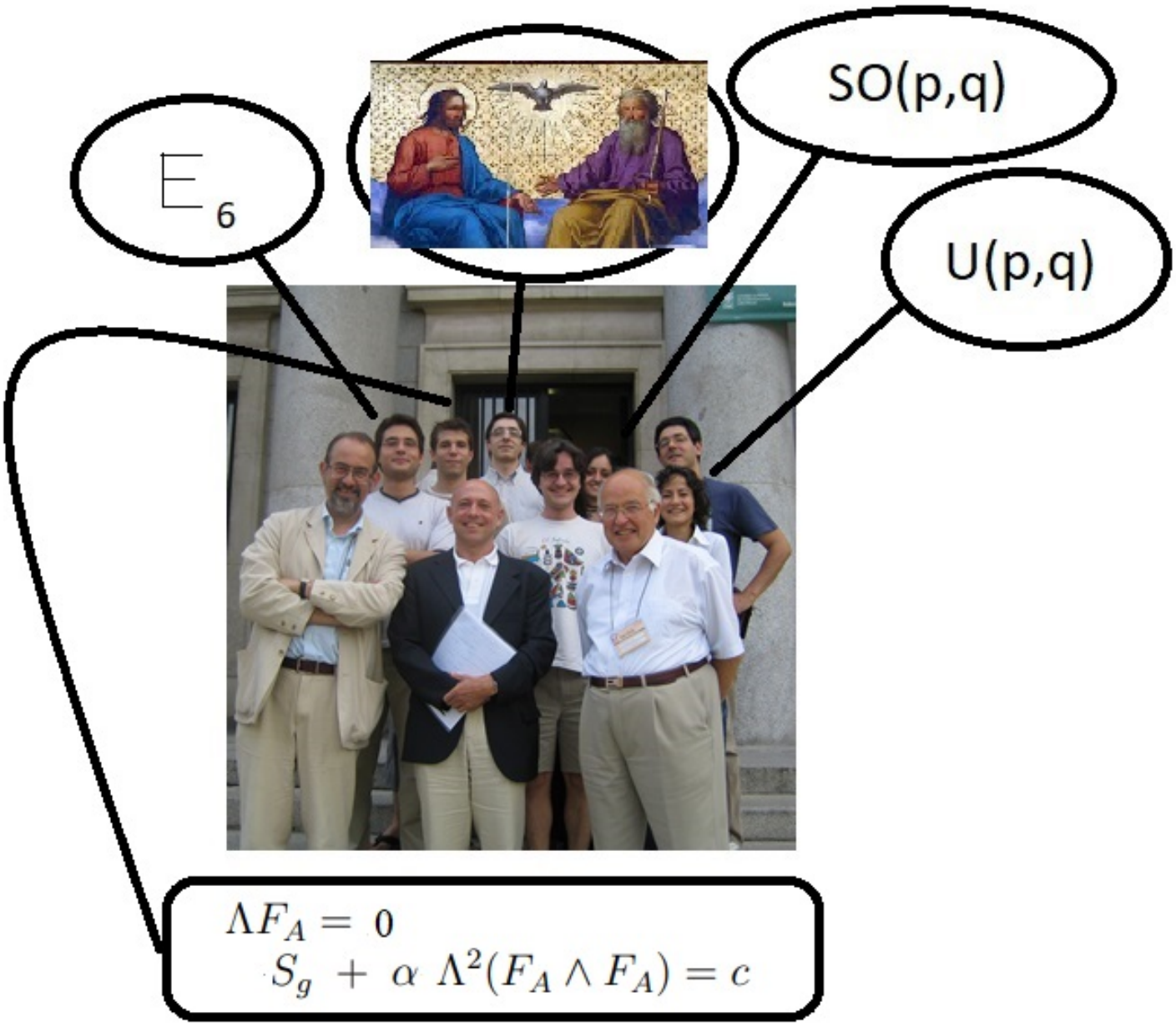
A family picture



A family picture



A family picture



The Green-Schwarz anomaly cancellation equation

$$dH = \operatorname{tr} R_\nabla \wedge R_\nabla - \operatorname{tr} F_A \wedge F_A$$

- M. Green and J. Schwarz, Anomaly cancellations in supersymmetric D = 10 gauge theory and superstring theory, Physics Letters B **149** (1984).

The Hull-Strominger system

Let (X, Ω) be a Calabi-Yau manifold of complex dimension n. Let V_0, V_1 holomorphic vector bundles over X such that

$$ch_2(V_0) = ch_2(V_1).$$

$$F_{h_0} \wedge \omega^{n-1} = 0 \qquad F_{h_1} \wedge \omega^{n-1} = 0$$

$$d(\|\Omega\|_\omega \omega^{n-1}) = 0 \qquad dd^c\omega - \operatorname{tr} F_{h_0} \wedge F_{h_0} + \operatorname{tr} F_{h_1} \wedge F_{h_1} = 0$$

- Strominger, Nucl. Phys. B 274 (1986).
- Hull, Turin 1985 Proc. (1986).

- $H = d^c\omega \neq 0 \Rightarrow \omega$ non-Kähler,
- Implies a refined cohomological equation $ch_2(V_0) = ch_2(V_1) \in H^{2,2}_{BC}(X)$

$$H^{p,q}_{BC}(X) = \frac{\ker d : \Omega^{p,q}(X, \mathbb{C}) \longrightarrow \Omega^{p+q+1}(X, \mathbb{C})}{\operatorname{Im} dd^c : \Omega^{p-1,q-1}(X, \mathbb{C}) \longrightarrow \Omega^{p,q}(X, \mathbb{C})}$$

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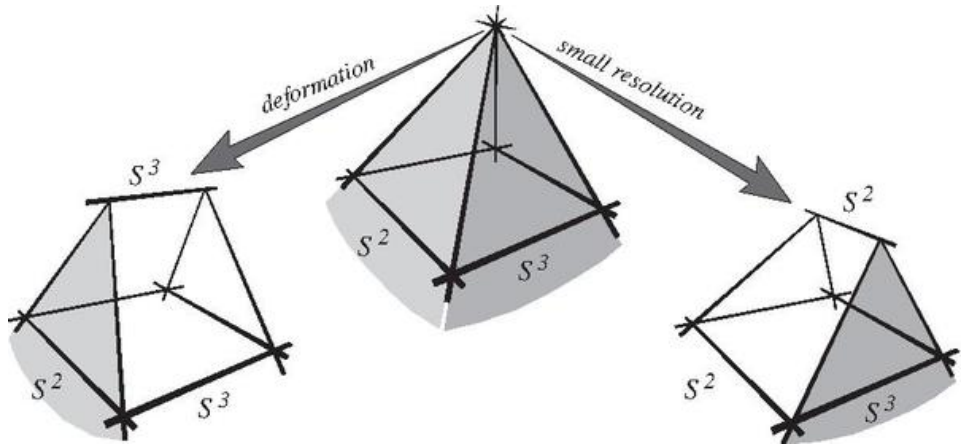
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The mathematical study of the **Hull-Strominger system** was initiated by Fu, Li, Tseng, and Yau more than 15 years ago, motivated by the geometrization of transitions and flops for Calabi-Yau threefolds.

- Li, Yau, JDG 70 (2005).
- Fu, Yau, JDG 78 (2008).
- Fu, Tseng, Yau, CMP 289 (2009).



Reid's Fantasy for algebraic Calabi-Yau threefolds:

There should exist a 'master moduli space' associated to the manifolds $\sharp_{k\geqslant 2}(S^3 \times S^3)$, such that every algebraic Calabi-Yau threefold arises as boundary phenomena for elements in this family.

- M. Reid, The moduli space of 3-folds with K=0 may nevertheless be irreducible, Math. Ann. **278** (1987).

A growing family

(0,2) Mirror Symmetry

$$(X, V) \leftrightarrow (X', V')$$

- Álvarez-Cónsul, De Arriba de la Hera, Garcia-Fernandez, arXiv:2012.01851 (2020).



Andoni De Arriba

Equivariant holomorphic $\mathbb{C}^*$-gerbes

$$B\mathbb{C}^* \to \mathcal{G} \to G \curvearrowright (P, \mathcal{L})$$

- Álvarez-Cónsul, Garcia-Fernandez, Tellez, ongoing.



Roberto Tellez

(Strong) Yau's Conjecture for Hull-Strominger

Let (X, Ω) be a Calabi-Yau manifold of complex dimension 3. Let V_0, V_1 holomorphic vector bundles over X such that

$$ch_2(V_0) = ch_2(V_1) \in H^{2,2}_{BC}(X).$$

Assume that there is a balanced metric ω_0 on X with balanced class $\mathfrak{b} = [\omega_0^2] \in H^{2,2}_{BC}(X)$ such that V_0, V_1 are $\mathfrak{b}$-slope polystable and

$$c_1(V_j) \cdot \mathfrak{b} = 0, \qquad j = 0, 1.$$

Then, (X, Ω, V_0, V_1) admits a solution of the Hull-Strominger system:

$$F_{h_0} \wedge \omega^2 = 0 \qquad F_{h_1} \wedge \omega^2 = 0$$

$$d(\|\Omega\|_\omega \omega^2) = 0 \qquad dd^c\omega - \operatorname{tr} F_{h_0} \wedge F_{h_0} + \operatorname{tr} F_{h_1} \wedge F_{h_1} = 0$$

with balanced class

$$[\|\Omega\|_\omega \omega^2] = \mathfrak{b} \in H^{2,2}_{BC}(X).$$

- Yau, Fourth Simons Workshop in Mathematics and Physics 2006, Stony Brook.

Nonetheless, the recent moment map interpretation of the Hull-Strominger system suggests the existence of obstructions beyond the balanced property of (X, Ω) and the slope stability of V_0 and V_1.

$$\int_X (d\xi - 2\,\mathrm{tr}\, s_0 F_{A_0} + 2\,\mathrm{tr}\, s_1 F_{A_1}) \wedge \|\Omega\|_\omega \omega^{n-1}.$$

- Garcia-Fernandez, Rubio, Shahbazi, Tipler, Proc. London Math.Soc. (2022).
- Garcia-Fernandez, Rubio, Tipler, JDG, to appear

A unifying principle

Something I learnt during my PhD ...

Complicated equations in mathematical physics can be often recasted in terms of simpler equations in a more intricate space.

$$\Lambda F_H = \lambda\,\mathrm{Id} \qquad \leftrightsquigarrow \qquad iF_h + \frac{1}{2} * (|\phi|^2 - \tau) = 0.$$

The Geometry of the Vortex Equation

Oscar García–Prada,
Wolfson College, Oxford.

Thesis submitted for the degree of D. Phil., August 1991.

Abstract

We first prove a G-invariant version of the theorem of Donaldson, Uhlenbeck and Yau relating the stability of a holomorphic bundle over a compact Kähler manifold to the existence of a Hermitian–Yang–Mills connection.

Let (ω, h_0, h_1) a solution of the anomaly equation on (X, V_0, V_1)

$$dd^c\omega - \operatorname{tr} F_{h_0} \wedge F_{h_0} + \operatorname{tr} F_{h_1} \wedge F_{h_1} = 0.$$

Consider the bundle

$$\mathcal{Q} = T^{1,0} \oplus End\ V_0 \oplus End\ V_1 \oplus T^*_{1,0},$$

with Dolbeault operator and pairing

$$\begin{aligned}\bar\partial_0(V + r + t + \xi) = \bar\partial V + i_V F_{h_0} + i_V F_{h_1} + \bar\partial_{V_0} r + \bar\partial_{V_1} t \\ + \bar\partial\xi - i_V(2i\partial\omega) - 2\operatorname{tr} F_{h_0} r + 2\operatorname{tr} F_{h_1} t\end{aligned}$$

$$\langle V + r + t + \xi, V + r + t + \xi\rangle = \xi(V) - \operatorname{tr} r^2 + \operatorname{tr} t^2$$

Lemma

The previous formulae define a holomorphic orthogonal bundle $(\mathcal{Q}, \langle , \rangle)$

$$\langle , \rangle \colon \mathcal{Q} \otimes \mathcal{Q} \to \mathcal{O}_X$$

Assume now that (ω, h_0, h_1) is a solution of the Hull-Strominger system on (X, V_0, V_1). In the corresponding bundle

$$\mathcal{Q} = T^{1,0} \oplus End\ V_0 \oplus End\ V_1 \oplus T^*_{1,0},$$

we define a pseudo-Hermitian metric

$$\mathbf{G} = \begin{pmatrix} g & 0 & 0 & 0 \\ 0 & \mathrm{tr}_{V_0} & 0 & 0 \\ 0 & 0 & -\mathrm{tr}_{V_1} & 0 \\ 0 & 0 & 0 & \frac{1}{4}g^{-1} \end{pmatrix}.$$

Proposition (GF, Gonzalez-Molina, '22)

Assume that (X, Ω, V_0, V_1) admits a solution of the Hull-Strominger system (ω, h_0, h_1). Then,

$$F_{\mathbf{G}} \wedge \omega^{n-1} = 0.$$

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- **Analogy:** (X, ω) Kähler-Einstein $\Rightarrow$ TX admits Hermitian-Einstein metric ($\Rightarrow$ TX is $[\omega]$-polystable).
- The explicit metric in the previous construction is **pseudo-Hermitian**

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Hence, there is no direct way to use Donaldson-Uhlenbeck-Yau to relate the Hull-Strominger system to a GIT stability condition for the holomorphic orthogonal bundle $(\mathcal{Q}, \langle , \rangle)$.

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- Inspired by a physical result by De La Ossa, Larfors, and Svanes (perturbation theory).
  - De la Ossa, Larfors, Svanes, Geometry and Physics: Volume II: A Festschrift in honour of N. Hitchin (2018).
- $F_{\mathbf{G}} \wedge \omega^{n-1} = 0$ strictly weaker than the Hull-Strominger system.

- R. Gonzalez Molina, *Bismut-Hermitian-Einstein metrics which are not Bismut flat*, to appear.



Gonzalez Molina

Futaki Invariants

Lemma (GF, Gonzalez-Molina, '22)

Let X be a compact complex manifold of dimension n, $\mathcal{Q}$ a holomorphic vector bundle over X, and $\mathfrak{b} \in H^{n-1,n-1}_{BC}(X)$. Then, the map

$$\mathcal{F}_{\mathfrak{b}} : H^0(X, \operatorname{End} \mathcal{Q}) \longrightarrow \mathbb{C}$$
$$\varphi \mapsto \int_X \operatorname{tr}(\varphi F_{\mathbf{G}}) \wedge \nu$$

defines a character of the Lie algebra $H^0(X, \operatorname{End} \mathcal{Q})$, which does not depend on the representative ν of $\mathfrak{b} = [\nu]$ and neither on the choice of a pseudo-Hermitian metric $\mathbf{G}$ on $\mathcal{Q}$.

In particular, $\mathcal{F}_{\mathfrak{b}} = 0$ if there exists a pseudo-Hermitian metric $\mathbf{G}$ on $\mathcal{Q}$ and a balanced Hermitian metric ω on X, with $\mathfrak{b} = [\omega^{n-1}]$, solving

$$F_{\mathbf{G}} \wedge \omega^{n-1} = 0.$$

Proposition (GF, Rubio, Tipler '20)

Let X be a compact complex manifold endowed with holomorphic vector bundles V_0, V_1 such that

$$ch_2(V_0) = ch_2(V_1) \in H^{2,2}_{BC}(X).$$

Then, there exists a canonical family of holomorphic orthogonal bundles $\{(\mathcal{Q}_\mathfrak{s}, \langle , \rangle)\}_{\mathfrak{s}\in\mathfrak{S}}$ given by holomorphic extensions ($A_{V_0\oplus V_1}$ Atiyah algebroid)

$$0 \longrightarrow T^*_{1,0} \longrightarrow \mathcal{Q}_\mathfrak{s} \longrightarrow A_{V_0\oplus V_1} \longrightarrow 0 \tag{1}$$

where $\mathfrak{S}$ is an affine space, modelled on the image of the natural map

$$\partial\colon H^{1,1}_A(X,\mathbb{R}) \to H^{2,1}_{\bar\partial}(X).$$

Consequently, if X satisfies the $\partial\bar\partial$-Lemma, then $\mathfrak{S} = \{\bullet\}$.

- Garcia-Fernandez, Rubio, Tipler, TAMS (2020).

Proof: classification of holomorphic Courant algebroids of string type.

Theorem (GF, Gonzalez-Molina, '22)

Let (X, Ω) be a Calabi-Yau manifold endowed with a holomorphic vector bundles V_0, V_1 such that

$$ch_2(V_0) = ch_2(V_0) \in H^{2,2}_{BC}(X).$$

Assume that (X, Ω, V_0, V_1) admits a solution of the Hull-Strominger system (ω, h_0, h_1) with balanced class

$$\mathfrak{b} = [\|\Omega\|_\omega \omega^{n-1}] \in H^{n-1,n-1}_{BC}(X)$$

Then, there exists $\mathfrak{s} \in \mathfrak{S}$ such that the associated *stringy Futaki invariant* is identically zero $\mathcal{F}^{\mathfrak{s}}_{\mathfrak{b}} \equiv 0$

$$\mathcal{F}^{\mathfrak{s}}_{\mathfrak{b}} : H^0(X, \mathrm{End}\ \mathcal{Q}_{\mathfrak{s}}) \longrightarrow \mathbb{C}.$$

This is the first known obstruction to the Hull-Strominger system, beyond the balanced property of X and the slope polystability of V_0, V_1.

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- In order to disprove (strong) Yau's Conjecture, it is sufficient to find V_0, V_1 $\mathfrak{b}$-slope polystable such that $0 \neq \mathcal{F}^{\mathfrak{s}}_{\mathfrak{b}}$, $\forall \mathfrak{s} \in \mathfrak{S}$.
- For X a $\partial\bar{\partial}$-manifold, if suffices to check that $0 \neq \mathcal{F}^{\{\bullet\}}_{\mathfrak{b}}$.

Example: Consider the one parameter family of pairs of real Lie algebras and cx. structures $(\mathfrak{g}_b, J_b)$, determined by ($\{\omega_i\}_{i=1,2,3}$ basis of $(\mathfrak{g}_b)^*_{1,0}$)

$$d\omega_1 = d\omega_2 = 0, \quad d\omega_3 = \omega_{12} + \omega_{1\bar{1}} + b\omega_{1\bar{2}} - \omega_{2\bar{2}}, \qquad b \in \mathbb{R}.$$

Given $(m, n, p) \in \mathbb{Z}^3 \backslash \{0\}$, define hermitian holomorphic line bundles $V_0^{m,n,p}$ on the Calabi-Yau nilmanifold $(X_{b,\Gamma}, \Omega) = (\Gamma \backslash G_b, J_b, \omega_{123})$ by

$$F = \pi(m(\omega_{1\bar{1}} - \omega_{2\bar{2}}) + n(\omega_{1\bar{2}} + \omega_{2\bar{1}}) + ip(\omega_{1\bar{2}} - \omega_{2\bar{1}})).$$

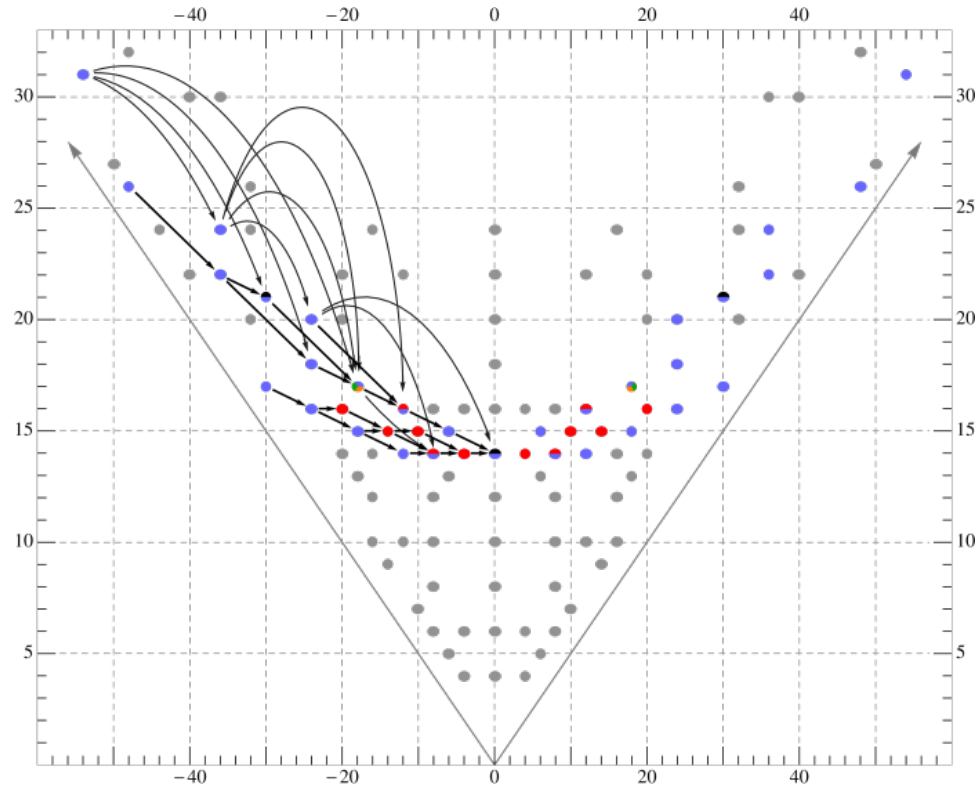
Then, $ch_2(V_0^{m,n,p}) = 0 \in H^{2,2}_{BC}(X_{b,\Gamma})$.

Proposition (GF, Gonzalez-Molina, '22)

Let $(X_{b,\Gamma}, \Omega, V_0^{m,n,p})$ as above. Then, $\exists\, \mathfrak{b} \in H^{2,2}_{BC}(X_{b,\Gamma})$ balanced class and $\mathfrak{s} \in \mathfrak{S} = \mathfrak{S}(V_0^{m,n,p})$ such that $c_1(V_0^{m,n,p}) \cdot \mathfrak{b} = 0$ and

$$0 \neq \mathcal{F}^{\mathfrak{s}}_{\mathfrak{b}} : H^0(X, \operatorname{End} \mathcal{Q}_{\mathfrak{s}}) \longrightarrow \mathbb{C}.$$

In the case of projective Calabi-Yau threefolds, this provides a new tool to detect the failure of the geometrization of conifold transitions via the Hull-Strominger system ($\sim$ *heterotic Calabi-Yau web*)



Harmonic metrics

The solution of the Hemitian-Einstein equation we have constructed on the holomorphic orthogonal bundle $(\mathcal{Q}, \langle , \rangle)$ is **pseudo-Hermitian**

$$F_{\mathbf{G}} \wedge \omega^{n-1} = 0.$$

No direct way to use Donaldson-Uhlenbeck-Yau to relate the Hull-Strominger system to a GIT stability condition for $(\mathcal{Q}, \langle , \rangle)$.

As a matter of fact, the naive stability condition does not work

$$T^*_{1,0} \hookrightarrow \mathcal{Q}.$$

- Candelas, Horowitz, Strominger, Witten, Vacuum Configurations for Superstrings, Nucl. Phys. B 258 (1) (1985) 46--74.

Motivated by a hyperKähler moment map construction, we introduce the following notion of harmonic metric:

Definition (GF, Gonzalez-Molina, '22)

Assume that (X, Ω, V_0, V_1) admits a solution (ω, h_0, h_1) of the Hull-Strominger system. Consider the associated holomorphic vector bundle $\mathcal{Q}$, with pseudo-Hermitian metric $\mathbf{G}$ and associated Chern connection $D^{\mathbf{G}}$.

We say that a positive definite Hermitian metric $\mathbf{H}$ is a **harmonic metric for the Hull-Strominger system** if

$$(\nabla^{\mathbf{H}})^* \Psi + i_{\theta^\sharp_\omega} \Psi = 0,$$

where we use the unique decomposition $D^{\mathbf{G}} = \nabla^{\mathbf{H}} + \Psi$.

Theorem (GF, Gonzalez-Molina, '22)

Assume that (X, Ω, V_0, V_1) admits a solution (ω, h_0, h_1) of the Hull-Strominger system with balanced class

$$\mathfrak{b} \in H^{n-1,n-1}_{BC}(X).$$

Consider the associated holomorphic vector bundle $\mathcal{Q}$, with pseudo-Hermitian metric $\mathbf{G}$ and Chern connection $D^{\mathbf{G}}$. Assume that this solution admits a harmonic metric.

Then, for every proper isotropic coherent subsheaf $\mathcal{F} \subset \mathcal{Q}$ which is preserved by $D^{\mathbf{G}}$, one has

$$\deg_{\mathfrak{b}} \mathcal{F} \leqslant 0.$$

Proof: decomposing $D^{\mathbf{G}} = D^{\mathbf{H}} + \phi$, one has

$$(F_{\mathbf{H}} + \tfrac{1}{2}[\phi \wedge \phi^{*_{\mathbf{H}}}]) \wedge \omega^{n-1} = 0,$$

and the proof follows as in Hitchin-Simpson's picture for Higgs bundles.

Example: Consider the Iwasawa manifold $X = H_{\mathbb{C}}/\Gamma$ where

$$H_{\mathbb{C}} = \left\{ \begin{pmatrix} 1 & z_2 & z_3 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} \mid z_i \in \mathbb{C} \right\}$$

and $\Gamma \subset \mathbb{C}$ are the Gaussian matrices (with entries in $\mathbb{Z}[i]$). There is a holomorphic global frame

$$\omega_1 = dz_1, \omega_2 = dz_2, \omega_3 = dz_3 - z_2 dz_1.$$

Given $(m_j, n_j, p_j) \in \mathbb{Z}^3 \backslash \{0\}$, $j = 0, 1$ define hermitian holomorphic line bundles V_j

$$F_{h_j} = \pi(m_j(\omega_{1\bar{1}} - \omega_{2\bar{2}}) + n_j(\omega_{1\bar{2}} + \omega_{2\bar{1}}) + ip_j(\omega_{1\bar{2}} - \omega_{2\bar{1}})).$$

Then, $ch_2(V_0) = ch_2(V_1) = 0 \in H^{2,2}_{BC}(X)$.

Proposition (GF, Gonzalez-Molina, '22)

Let $(X, V_0.V_1)$ as above and assume $m_0^2 + n_0^2 + p_0^2 \neq m_1^2 + n_1^2 + p_1^2$ and $c_1(V_0) \cdot c_1(V_1) = 0$. Then, for any

$$L_0, L_1 \in Pic^0_X, \qquad \mathfrak{s} \in \mathfrak{S} = \mathfrak{S}(V_0 \otimes L_0, V_1 \otimes L_1)$$

there exists a solution of the Hull-Strominger which admits a harmonic metric.

Example: Consider the Iwasawa manifold $X = H_{\mathbb{C}}/\Gamma$ where

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Explicitly, in these examples

$$\omega = i\pi^2(m_0^2 + n_0^2 + p_0^2 - m_1^2 - n_1^2 - p_1^2)(\omega_{1\bar{1}} + \omega_{2\bar{2}} + \omega_{3\bar{3}})$$

$$\mathbf{H} = \begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{for} \quad \mathcal{Q} \cong (TX \oplus \mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{C},$$

where $g = \omega(, J)$, and also

$$\nabla^{\mathbf{H}} = \begin{pmatrix} \nabla^- & 0 & ig^{-1}F_{h_1} \\ 0 & d & 0 \\ iF_{h_1} & 0 & d \end{pmatrix}, \qquad \Psi = \begin{pmatrix} 0 & -ig^{-1}F_{h_0} & 0 \\ iF_{h_0} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Problem

Does any solution of the Hull-Strominger system admit a **harmonic metric H**?

$$(\nabla^{\mathbf{H}})^{*}\Psi + i_{\theta^{\sharp}_{\omega}}\Psi = 0.$$

Happy Birthday!

HARMONIC METRICS FOR THE HULL-STROMINGER SYSTEM AND STABILITY

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Dedicated to Oscar García-Prada on the occasion of his sixtieth birthday.

ABSTRACT. We investigate stability conditions related to the existence of solutions of the Hull-Strominger system with prescribed balanced class. We build on recent work by the authors, where the Hull-Strominger system is recasted using indefinite Hermitian-Einstein metrics on holomorphic Courant algebroids. Our main development is a notion of *harmonic metric* for the Hull-Strominger system, motivated by an infinite-dimensional hyperKähler moment map and related to a numerical stability condition, which we expect to exist generically for families of solutions. We illustrate our theory with an infinite number of continuous families of examples on the Iwasawa manifold.