Stratifications on the Nilpotent Cone of the Hitchin map

Ronald A. Zúñiga-Rojas CIMPA–UCR

[jww Peter B. Gothen]

ICMAT 2022:
Moduli Spaces and Geometric Structures, Conference in honour of Óscar García–Prada

EMat Escuela de Matemática

Introduction

Here, we study the stratification of the Nilpotent Cone given by the Morse Flow, the Downward Morse Flow actually, and its relation to another strata, the Shatz stratification. The results presented here complement those of [1] and [11]. We present a filtration that describes the Nilpotent Cone in terms of the Downward Morse Flow, for rank two and rank three cases. Furthermore, there is possible to set the particular case of filtration for a $(1,\dots,1)$-VHS in general rank r.

Preliminaries

The Hitchin Map and The Nilpotent Cone
Here, we are supposing $\mathrm{GCD}(r,d)=1$. So, the moduli space of Hitchin pairs, $\mathcal{M}_L(r,d)$, is a non-compact, smooth complex manifold of dimension

$$\dim_{\mathbb{C}}\big(\mathcal{M}_L(r,d)\big)=(r^2-1)\mathrm{deg}(L).$$

$\mathcal{M}_L(r,d)$ is also a Riemannian manifold with a complete hyperKähler metric, and there is a proper map, the so-called Hitchin map defined by:

$$\begin{array}{rcl}\chi:\mathcal{M}_L(r,d) & \longrightarrow & H^0(X,L)\oplus\cdots\oplus H^0(X,L^r)\\ [(E,\Phi)] & \longmapsto & \det(\Phi)\end{array} \tag{0.1}$$

The Hitchin map is proper, and it is also an algebraically completely integrable Hamiltonian system with respet to the symplectic holomorphic form Ω, with a generic fibre which is a Prym variety corresponding to the espectral cover of X at the image point.
Finally, recall also that the set

$$\chi^{-1}(0):=\big\{[(E,\Phi)]\in\mathcal{M}_L(r,d):\quad \chi(\Phi)=0\big\}$$

is known as the Nilpotent Cone, and has been described by Hitchin [6], Hausel [4], among others, as one of the most important fibres of the Hitchin map, and the most singular at the same time.
The Hitchin map is widely studied and descripted by Hausel [4] and [5]. Among his results, the most relevant is the following assertion:

Theorem (Hausel [4, Theorem 4.4.2]). *The Downward Morse Flow of $\mathcal{M}_L(r,d)$ coincides with the Nilpotent Cone:*

$$\chi^{-1}(0)\cong\bigcup_{\lambda}D^M_{\lambda}.$$

Hence,

$$[(E,\Phi)]\in\chi^{-1}(0)\Leftrightarrow\exists\lim_{z\to\infty}z\cdot(E,\Phi)\in\mathcal{M}_L(r,d).$$

Result 1

Rank Two Hitchin Pairs in the Nilpotent Cone
From the last theorem, we can conclude our own general results for the Hitchin pairs in the Nilpotent Cone. First, for rank two Hitchin pairs $(E,\Phi)\in\mathcal{M}_L(2,d)$, we have:

Theorem. *Let $[(E,\Phi)]\in\chi^{-1}(0)$ be a Hitchin pair with $\mathrm{rk}(E)=2$. Then, there is a filtration*

$$0=E_0\subseteq E_1\subseteq E_2=E$$

such that

$$\Phi(E_j)\subset E_{j-1}\otimes L$$

and

$$(E_\infty,\Phi_\infty):=\lim_{z\to\infty}(E,z\cdot\Phi)=\left(L_1\oplus L_2,\begin{pmatrix}0&0\\ \varphi&0\end{pmatrix}\right) \tag{0.2}$$

is a $(1,1)$-VHS where

$$L_j=E_j/E_{j-1}\quad\text{and}\quad\varphi:L_1\to L_2\otimes L.$$

Result 2:

Rank Three Hitchin Pairs in the Nilpotent Cone
We would like to say that the result is analogue for rank three Hitchin pairs $(E,\Phi)\in\mathcal{M}_L(3,d)$, but truth is that there is a bizard subcase where we must consider the image subsheaf of the k-Higgs field. So we get the following:

Theorem (Th. 2 [2]). *Let $[(E,\Phi)]\in\chi^{-1}(0)$ be a Hitchin pair with $\mathrm{rk}(E)=3$. Then:*

- *(a.) either there is a filtration*

$$0=E_0\subseteq E_1\subseteq E_2\subseteq E_3=0\quad\text{such that}\quad\Phi(E_{j+1})\subset E_j\otimes L\quad\text{and}$$

$$(E_\infty,\Phi_\infty):=\lim_{z\to\infty}(E,z\cdot\Phi)=\left(L_1\oplus L_2\oplus L_3,\begin{pmatrix}0&0&0\\ \varphi_1&0&0\\ 0&\varphi_2&0\end{pmatrix}\right) \tag{0.3}$$

  is a $(1,1,1)$-VHS where $L_j=E_j/E_{j-1}$ and $\varphi_j:L_j\to L_{j+1}\otimes L$

- *(b.) or, there is a filtration*

$$0=E_0\subseteq E_1\subseteq E_2\subseteq E_3=E\quad\text{such that}$$

  - *(b.1.) either*

$$(E_\infty,\Phi_\infty):=\lim_{z\to\infty}(E,z\cdot\Phi)=\left(V_1\oplus V_2,\begin{pmatrix}0&0\\ \varphi_{21}&0\end{pmatrix}\right) \tag{0.4}$$

    is a $(1,2)$-VHS where $V_j=E_j/E_{j+1}$ and $\varphi_{21}:V_1\to V_2\otimes L$ and where $\Phi(E_j)\subset E_{j+1}\otimes L$,

  - *(b.2.) or*

$$(E_\infty,\Phi_\infty):=\lim_{z\to\infty}(E,z\cdot\Phi)=\left(W_1\oplus W_2,\begin{pmatrix}0&0\\ \varphi_{21}&0\end{pmatrix}\right) \tag{0.5}$$

    is a $(2,1)$-VHS, depending on the rank of E_2, and depending also on some properties of Φ.

Result 3

General Rank
Suppose now that $[(E,\Phi)]\in\chi^{-1}(0)\subseteq\mathcal{M}_L(r,d)$ is a Hitchin pair of general rank r and degree d. Let $p\in\mathbb{N}$ be the least positive integer such that $\Phi^p=0$ and $\Phi^{p-1}\neq0$, and so let $N_j:=\ker(\Phi^j)\subset E$ be a sequence of subbundles of E $\forall j\in\{1,...,p\}$. We would like to conclude that there is a filtration

$$0=E_0\subseteq E_1\subseteq\dots\dots\subseteq E_{r-1}\subseteq E_p=E\quad\text{such that}\quad\Phi(E_j)\subset E_{j+1}\otimes L\quad\text{and}$$

$$(E_\infty,\Phi_\infty):=\lim_{z\to\infty}(E,z\cdot\Phi)=\left(\bigoplus_{j=1}^{p}V_j,\begin{pmatrix}0&\dots&\dots&\dots&0\\ \varphi_2&0&\dots&\dots&0\\ 0&\varphi_3&0&\dots&0\\ \vdots&\ddots&\ddots&\ddots&\vdots\\ 0&\dots&0&\varphi_p&0\end{pmatrix}\right) \tag{0.6}$$

where $V_j=E_j/E_{j-1}$ and $\varphi_j:V_j\to V_{j-1}\otimes L$, but this is not always true. Nevertheless, when $p=r\in\mathbb{N}$ (E,Φ) has a filtration

$$0=E_0\subseteq E_1\subseteq\dots\dots\subseteq E_{r-1}\subseteq E_r=E$$

and so, the limit (E_∞,Φ_∞) will be a $(1,\dots,1)$-VHS (see Th. 1 [2]).

References

[1] P.B. Gothen and R.A. Zúñiga–Rojas, *Stratifications on the moduli space of Higgs bundles*, Portugaliae Mathematica, EMS **74** (2017), 127–148.

[2] P.B. Gothen and R.A. Zúñiga–Rojas, *Stratifications on the Nilpotent Cone of the moduli space of Hitchin pairs*, Rev. Matemática Complutense, Springer **35** (2022), 311–321.

[3] G. Harder and M.S. Narasimhan, *On the Cohomology Groups of Moduli Spaces of Vector Bundles on Curves*, Math. Ann. **212** (1975), 215–248.

[4] T. Hausel, "Geometry of Higgs Bundles", Ph.D. Thesis, Cambridge, United Kingdom, 1998.

[5] T. Hausel, *Global topology of the Hitchin system*, Handbook of moduli. Vol. II, Adv. Lect. Math. (ALM), vol. 25, Int. Press, Somerville, MA, 2013, pp. 29–69.

[6] N.J. Hitchin, *The Self-Duality Equations on a Riemann Surface*, Proc. London Math. Soc. **55** (1987), no. 3, 59–126.

[7] N.J. Hitchin, *Stable bundles and integrable systems*, Duke Math. J. **54** (1987), 91–114.

[8] M.S. Narasimhan and C.S. Seshadri, "Stable and Unitary Vector Bundles on a Compact Riemann Surface" *The Annals of Mathematics*, Second Series Vol. **82** (1965), pp. 540-567.

[9] S.S. Shatz, *The Decomposition and Specialization of Algebraic Families of Vector Bundles*, Compositio Mathematica **35** (1977), no. 2., 163–187.

[10] C.T. Simpson, *Higgs Bundles and Local Systems*, Inst. Hautes Études Sci. Math. Publ. **75** (1992), 5–95.

[11] R.A. Zúñiga–Rojas, *Estratificações no espaço móduli dos fibrados de Higgs*, Boletim da SPM, Número Especial, ENSPM 2014, Sessão Alunos de Doutoramento, Lisboa (2016), 129–133.

[12] R.A. Zúñiga–Rojas, "Homotopy Groups of the Moduli Space of Higgs Bundles", Ph.D. Thesis, Porto, Portugal, 2014.

Contact Information

Name: Ronald A. Zúñiga-Rojas **e-mail** ronald.zunigarojas@ucr.ac.cr
website https://paginas.cimpa.ucr.ac.cr/ronaldz/index.php/cv **phone** +506 2511-3450 / +506 8372-7737