Toda lattice with opposite sign on Riemannian manifolds and the variational method

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1 Toda lattice with opposite sign

Let X be a connected Riemann surface with the canonical bundle $K_X \to X$. We take a square root $K_X^{1/2} \to X$ and we define a holomorphic vector bundle $E \to X$ of rank r as $E \coloneqq K_X^{(r-1)/2} \oplus K_X^{(r-3)/2} \oplus \cdots \oplus K_X^{-(r-3)/2} \oplus K_X^{-(r-1)/2}$. We take a diagonal Hermitian metric $h = (h_1, \dots, h_r)$ on E with curvature F_h such that $\det(h) = 1$. We also take a Kähler form ω_X. We denote by Λ_{ω_X} the dual of $\omega_X \wedge$, and by Δ_{ω_X} the geometric Laplacian. Let $v_1, \dots, v_r$ be vectors of $V \coloneqq \{x = (x_1, \dots, x_r) \in \mathbb{R}^r \mid x_1 + \cdots + x_r = 0\}$ defined as $v_j \coloneqq e_{j+1} - e_j$ $(j = 1, \dots, r-1)$, $v_r \coloneqq e_1 - e_r$, where we denote by $e_1, \dots, e_r$ the canonical basis of $\mathbb{R}^r$. Let $(\cdot, \cdot)$ be the canonical inner product of $\mathbb{R}^r$. For each $q \in H^0(K_X^r)$, we define the following PDE for V-valued function $\xi : X \to V$, which is called the Toda lattice with opposite sign [2]:

$$\Delta_{\omega_X} \xi + \sum_{j=1}^{r} 4k_j e^{(v_j, \xi)} v_j = -2\sqrt{-1}\Lambda_{\omega_X} F_h, \tag{1}$$

where $\mathbb{R}^r$ is identified with diagonal matrices, and we denote by $k_1, \dots, k_r$ the positive functions defined by $k_j \coloneqq |1|^2_{h,\omega_X}$ $(j = 1, \dots, r-1)$, $k_r \coloneqq |q|^2_{h,\omega_X}$. Note that 1 (resp. q) is considered to be a K_X^{-1} (resp. K_X^{r-1})-valued holomorphic 1-form. PDE (1) is the Hermitian-Einstein equation for diagonal harmonic metrics on a cyclic Higgs bundle [1].

2 Toda lattice with opposite sign on Riemannian manifolds

Let (M, g_M) be a Riemannian manifold with geometric Laplacian Δ_{g_M}. Let $a_1, \dots, a_r : M \to \mathbb{R}_{\geq 0}$ be non-negative, not-globally zero smooth functions and $w : M \to V$ a V-valued smooth function. We generalize (1) to a PDE on (M, g_M):

$$\Delta_{g_M} \xi + \sum_{j=1}^{r} a_j e^{(v_j, \xi)} v_j = w. \tag{2}$$

Theorem 1. (M. [3, 5]) Suppose that M is compact, and that for each $j = 1, \dots, r$, $a_j^{-1}(0)$ is a measure zero and $\log a_j$ is integrable. Then there uniquely exists a smooth solution of (2). Moreover, if there exists a Riemannian foliation on (M, g_M) and if $a_1, \dots, a_r, w$ are all basic functions concerning the foliation, then the unique solution of (2) is a basic function.

From Theorem 1, we obtain approximations of harmonic bundles that are not globally harmonic, but are locally harmonic on some open subsets:

Example 2. (Perturbation of the Hermitian-Einstein equation) Let $T = (T_1, \dots, T_r)$ be $\mathbb{R}^r$-valued real current on X satisfying $T_1 + \cdots + T_r = 0$. We consider equation (1) perturbed by T:

$$\Delta_{\omega_X} \xi + \sum_{j=1}^{r} 4k_j e^{(v_j, \xi)} v_j = -2\sqrt{-1}\Lambda_{\omega_X} F_h + T \tag{3}$$

On an open subset $U \subseteq X$ such that $T \mid_U = 0$, (3) is the usual Hermitian-Einstein equation, and we obtain a harmonic bundle on U from a solution of (3). If X is compact and if T is smooth, then from Theorem 1, (3) has a unique solution for any T.

Example 3. (Cyclic Higgs bundles with not-holomorphic Higgs fields) Let q be a smooth, but not holomorphic section of $K_X^r \to X$. We consider (1) for not-holomorphic q. Suppose that there exists a local section $q' \in H^0(K_X^r \mid_U)$ satisfying $|q|^2 = |q'|^2$ on an open subset $U \subseteq X$. Then we obtain a harmonic bundle on U from a solution of (1) by replacing q with q'. If X is compact and if $\log |q|^2$ is integrable, then from Theorem 1, we have a unique solution of (1) for not-holomorphic q.

Example 4. (Three-dimensional Toda lattice with opposite sign combined with Example 2 and Example 3) Let X be a compact orbifold Riemann surface and $M \to X$ a subjective submersion from a 3-dimensional smooth compact manifold M. We can consider a 3D analogue of (1) on M, and we can also combine it with Example 2 and Example 3.

Example 5. (Toda lattice with opposite sign on higher-dimensional Kähler manifolds combined with Example 2 and Example 3) We can consider an analog of (1) on more general Kähler manifolds and we can also combine Example 2 and Example 3. Note, however, that for higher dimensions, the combination of the above examples makes it difficult to obtain the harmonic bundles from the vanishing Chern classes.

Theorem 6. (M.) Let (M, g_M) be a compact Riemannian manifold with smooth, possibly empty boundary ∂M. Then for any smooth $\eta : M \to V$, there exists a unique smooth family $(\xi_t)_{t \in [0,\infty)}$ such that

$$\frac{d\xi_t}{dt} + \Delta_{g_M} \xi_t + \sum_{j=1}^{r} a_j e^{(v_j, \xi_t)} v_j - w = 0 \text{ for all } t \in [0, \infty),$$
$$\xi_t \mid_{\partial M} = \eta \mid_{\partial M} \text{ for all } t \in [0, \infty),\ \xi_0 = \eta.$$

Remark 7. A similar result also holds for the Neumann condition.

Theorem 8. (M.) Let (M, g_M) be a compact Riemannian manifold with a non-empty smooth boundary ∂M. Then for any smooth $\eta : \partial M \to V$, there uniquely exists a solution $\xi : M \to V$ of (2) satisfying $\xi \mid_{\partial M} = \eta$.

3 More generalization and Donaldson's functional

For the proof of Theorem 1, we use the variational method inspired by the Kempf-Ness theorem for linear torus actions. For the relationship between the Kempf-Ness theorem for linear torus actions and the usual stability condition, see [4, Appendix]. Note that in our discussion essentially only Jensen's inequality and Poincaré inequality are used to evaluate the functional from below. This means that even if the Higgs field is not holomorphic, Donaldson's functional has a critical point under strong assumptions. Details will be discussed in [6].

References

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