Lie algebroid connections, twisted Higgs bundles and motives of moduli spaces

COMILLAS UNIVERSIDAD PONTIFICIA ICAI DEPARTAMENTO DE MATEMÁTICA APLICADA

David Alfaya

Comillas Pontifical University

joint work with **André Oliveira**

Definition (Lie Algebroid)

A ***Lie algebroid*** on Y is a triple $\mathcal{L} = (V, [\cdot,\cdot], \delta)$, where

- V vector bundle on Y,
- $[\cdot,\cdot]: V \otimes_{\mathbb{C}} V \to V$ $\mathbb{C}$-bilinear skew-symmetric Lie bracket, and
- $\delta: V \to T_Y$ vector bundle map called anchor.

Satisfying

- ***Jacobi identity***: $[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$.
- ***Leibniz rule***: $[u, fv] = f[u,v] + \delta(u)(f)v$.

We can associate to $\mathcal{L}$ its Chevalley-Eilenberg-de Rham complex $(\Omega^{\cdot}_{\mathcal{L}} \coloneqq \wedge^{\cdot} V^*, d_{\mathcal{L}})$:

$$d_{\mathcal{L}}(w)(v_1,\dots,v_{n+1}) = \sum_i (-1)^{i+1}\delta(v_i)\big(w(v_1,\dots,\widehat{v_i},\dots,v_{n+1})\big) + \sum_{i<j}(-1)^{i+j} w\big([v_i,v_j],v_1,\dots,\widehat{v_i},\dots,\widehat{v_j},\dots,v_{n+1}\big).$$

Definition (Lie Algebroid connection)

An **$\mathcal{L}$-connection** on Y is a pair $(E, \nabla_{\mathcal{L}})$ where E is a vector bundle on Y and

$$\nabla_{\mathcal{L}}: E \to E \otimes \Omega^1_{\mathcal{L}}$$

is a $\mathbb{C}$-linear map satisfying a version of Leibniz rule

$$\nabla_{\mathcal{L}}(fs) = f\nabla_{\mathcal{L}}(s) + s \otimes d_{\mathcal{L}}(f), \qquad f \in \mathcal{O}_X, \qquad s \in E.$$

We say that $(E, \nabla_{\mathcal{L}})$ is ***integrable*** if, moreover, $\nabla^2_{\mathcal{L}} = \nabla_{\mathcal{L}} \wedge \nabla_{\mathcal{L}} + d_{\mathcal{L}}(\nabla_{\mathcal{L}}) = 0$.

Some examples of integrable $\mathcal{L}$-connections

$\mathcal{L}$-connections provide a **unified framework** for studying multiple moduli spaces

$\mathcal{L} = (T_Y, [\cdot,\cdot] = 0, \delta = 0)$	**Higgs bundles**
$\mathcal{L} = (V, [\cdot,\cdot] = 0, \delta = 0)$	**V^*-twisted Higgs bundles**
$\mathcal{L} = (T_Y, [\cdot,\cdot]_{Lie}, \delta = Id)$	**Flat connections**
$\mathcal{L} = (T_Y(-D), [\cdot,\cdot]_{Lie}, \delta: T_Y(-D) \hookrightarrow T_Y)$	**Integrable meromorphic connections (poles on D)**
$\mathcal{L}_\lambda = (T_Y, \lambda[\cdot,\cdot]_{Lie}, \delta = \lambda Id)$	**Flat λ-connections (Hodge moduli spaces)**

Let X be a Riemann surface. Let $\mathcal{M}_{\mathcal{L}}(r,d)$ denote the ***moduli space of $\mathcal{L}$-connections*** of rank r and degree d on X. By [Tor12], there is a correspondence between $\mathcal{L}$-connections and $\Lambda_{\mathcal{L}}$-modules for a certain sheaf of rings of differential operators $\Lambda_{\mathcal{L}}$. Then, by [Sim94], the moduli space exists and it is a quasi-projective variety, but for general $\mathcal{L}$, **little was known about its geometry**. This work is a first step towards filling this gap.

Theorem (Invariance of motives and E-polynomials)

Let X be a smooth Riemann surface of genus $g \geq 2$. Let $\mathcal{L}$ and $\mathcal{L}'$ be Lie algebroids on line bundles L and L' of the same degree $d_L < 2-2g$. Suppose that r and d are coprime. Then

$$[\mathcal{M}_{\mathcal{L}}(r,d)] = [\mathcal{M}_{\mathcal{L}'}(r,d)] \in \widehat{K}(Var_{\mathbb{C}})$$

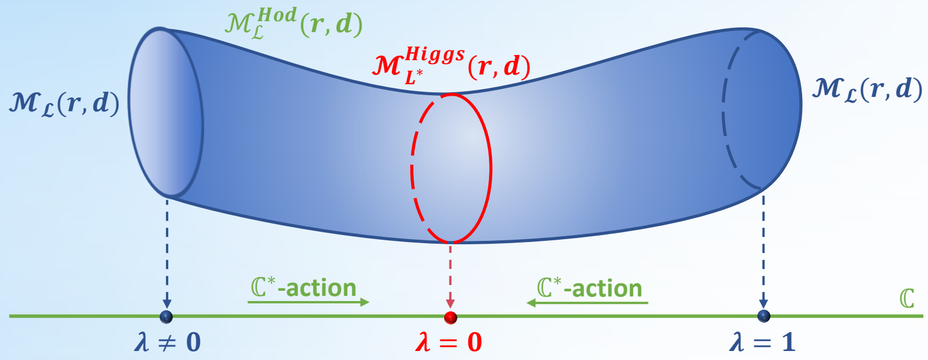
in the **Grothendieck ring of varieties**. Moreover, for any other d' coprime with r, then

$$E[\mathcal{M}_{\mathcal{L}}(r,d)] = E[\mathcal{M}_{\mathcal{L}'}(r,d')]$$

Similar results also hold for **Voevodsky motives**, **Chow motives**, **Chow rings**, **mixed Hodge structures** and for moduli spaces where the determinant is fixed instead of the degree.

Idea of the proof.

We prove that for each $\mathcal{L} = (L, [\cdot,\cdot], \delta)$ there exists a smooth variety $\pi: \mathcal{M}^{Hod}_{\mathcal{L}}(r,d) \to \mathbb{C}$ over $\mathbb{C}$, that we call **$\mathcal{L}$-Hodge moduli space**, with the following properties:



1) There is an action $\mathbb{C}^* \circlearrowright \mathcal{M}^{Hod}_{\mathcal{L}}(r,d)$.
2) The fixed point locus for the action is proper.
3) For each point p, there exists the limit $\lim_{t\to 0} t \cdot p$.

} ***Semiprojective variety***

4) The map π is a surjective $\mathbb{C}^*$-equivariant submersion for the scaling action on $\mathbb{C}$.
5) $\forall \lambda \neq 0$, $\pi^{-1}(\lambda) \cong \mathcal{M}_{\mathcal{L}}(r,d)$, but $\pi^{-1}(0) \cong \mathcal{M}^{Higgs}_{L^*}(r,d)$ is the **L^*-Higgs moduli**.

The variety $\mathcal{M}^{Hod}_{\mathcal{L}}(r,d)$ is built as the **moduli space of $(\lambda,\mathcal{L})$-connections**, i.e., triples (E,∇,λ) where $\lambda \in \mathbb{C}$ and $\nabla: E \to E \otimes \Omega^1_{\mathcal{L}}$ such that

$$\nabla(fs) = f\nabla(s) + \lambda s \otimes d_{\mathcal{L}}(f)$$

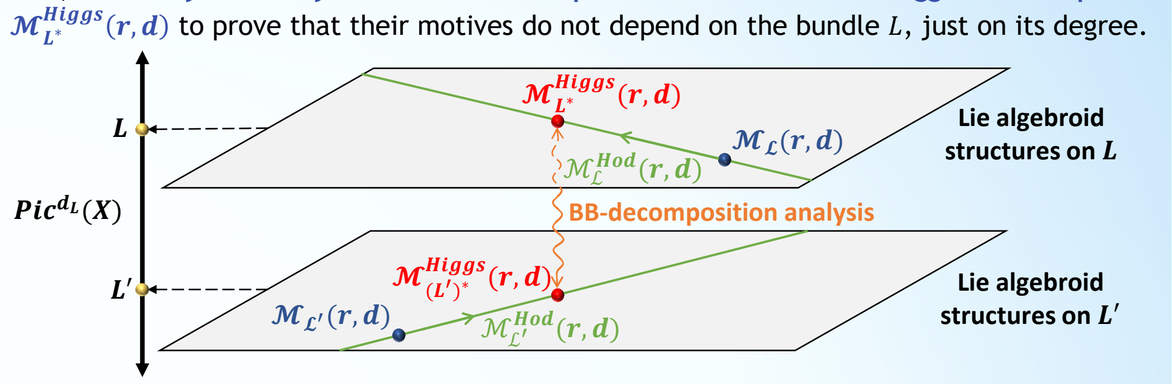
The $\mathbb{C}^*$-action is then $t \cdot (E,\nabla,\lambda) = (E, t\nabla, t\lambda)$. All fibres of the map $\pi(E,\nabla,\lambda) = \lambda$ are isomorphic but, for $\lambda = 0$, the map ∇ is $\mathcal{O}_X$-linear and, thus, a **$\Omega^1_{\mathcal{L}} = L^*$-twisted Higgs field**. We then prove the following theorem which, together with the previous construction, shows that $[\mathcal{M}_{\mathcal{L}}(r,d)] = \left[\mathcal{M}^{Higgs}_{L^*}(r,d)\right]$. Let $\mathbb{L} = [\mathbb{C}]$.

Theorem (Motives of semiprojective varieties)

If $\pi: \mathcal{M} \to \mathbb{C}$ is any smooth variety satisfying conditions (1)-(4) above, then, in $\widehat{K}(Var_{\mathbb{C}})$,

$$[\pi^{-1}(0)] = [\pi^{-1}(1)] \qquad \text{and} \qquad [\mathcal{M}] = \mathbb{L}[\pi^{-1}(0)]$$

Now, we **study the Bialynicki-Birula decomposition of the twisted Higgs moduli spaces** $\mathcal{M}^{Higgs}_{L^*}(r,d)$ to prove that their motives do not depend on the bundle L, just on its degree.



Definition (Bialynicki-Birula decomposition)

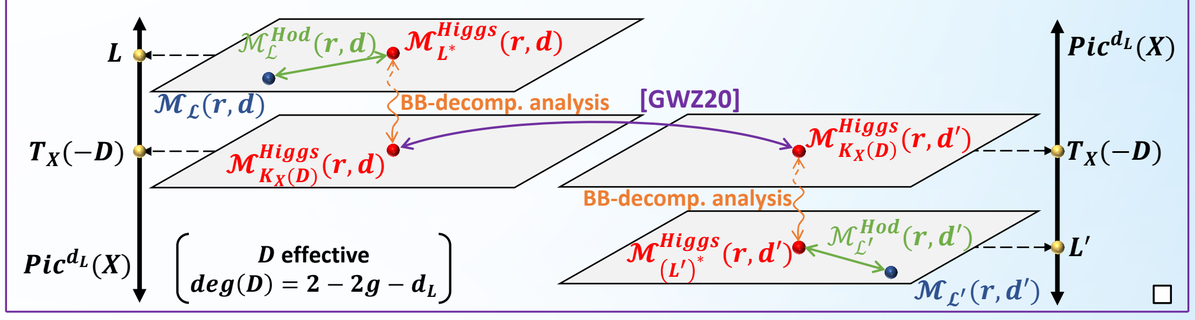
Let $\mathcal{M}$ be a smooth semiprojective variety. Let $\mathcal{M}^{\mathbb{C}^*} = \coprod_\mu F_\mu$ be the decomposition of the fixed point locus in connected components. Then there is a stratification

$$\mathcal{M} = \coprod_\mu U^+_\mu; \qquad U^+_\mu = \left\{p \in \mathcal{M} \mid \lim_{t\to 0} t \cdot p \in F_\mu\right\}.$$

Moreover, $U^+_\mu \to F_\mu$ is an algebraic affine fibration of rank N^+_μ. Thus, in $\widehat{K}(Var_{\mathbb{C}})$,

$$[\mathcal{M}] = \sum_\mu \mathbb{L}^{N^+_\mu}[F_\mu].$$

We show that for each L, L' of the same degree d_L, there exist isomorphisms between the components of the fixed point loci of $\mathcal{M}^{Higgs}_{L^*}(r,d)$ and $\mathcal{M}^{Higgs}_{(L')^*}(r,d)$ (they are **different, but isomorphic moduli spaces of chain bundles**). The corresponding ranks N^+_μ are independent from L, yielding the result. For E-polynomials, we can also change d using the independence result on d for the $K_X(D)$-twisted Higgs moduli in [GWZ20].



The previous geometric analysis also allows us to obtain the following results.

Theorem (Topology of the moduli space)

Let X be a smooth Riemann surface of genus $g \geq 2$. Let $\mathcal{L} = (L, [\cdot,\cdot], \delta)$ be a Lie algebroid such that L is a line bundle of degree $\deg(L) < 2-2g$. Then

- $\mathcal{M}_{\mathcal{L}}(r,d)$ is **non-empty**, **smooth** and **irreducible**.
- $\dim(\mathcal{M}_{\mathcal{L}}(r,d)) = 1 - r^2\deg(L)$.
- If $(g,r) \neq (2,2)$, $\pi_1(\mathcal{M}_{\mathcal{L}}(r,d)) = H_1(X,\mathbb{Z}) \cong \mathbb{Z}^{2g}$, and $\pi_2(\mathcal{M}_{\mathcal{L}}(r,d)) \cong \mathbb{Z}$.
- For $k = 3,\dots,2(g-1)(r-1)-2$, if $\mathcal{G}_r$ is the Gauge group or a rank r Hermitian vector bundle on X then $\pi_k(\mathcal{M}_{\mathcal{L}}(r,d)) \cong \pi_k(\mathcal{G}_r)$.

Theorem (Explicit motive computation in ranks 2 and 3)

Let X be a smooth Riemann surface of genus $g \geq 2$. Let $\mathcal{L} = (L, [\cdot,\cdot], \delta)$ be a Lie algebroid such that L is a line bundle of degree $\deg(L) = d_L < 2-2g$. In $\widehat{K}(Var_{\mathbb{C}})$, decompose $[X] = 1 + h^1(X) + \mathbb{L}$. Consider the λ-ring structure on $\widehat{K}(Var_{\mathbb{C}})$ induced by $\lambda^k([Y]) = [Sym^k(Y)]$. Let $P_X(z) = \sum_{n=0}^{2g} \lambda^n(h^1(X))\, z^n$. Then, in $\widehat{K}(Var_{\mathbb{C}})$, if d is coprime with the rank,

$$[\mathcal{M}_{\mathcal{L}}(2,d)] = \frac{\mathbb{L}^{-4d_L+4-4g}(P_X(1)P_X(\mathbb{L}) - \mathbb{L}^g P_X(1)^2)}{(\mathbb{L}-1)(\mathbb{L}^2-1)} + \mathbb{L}^{-3d_L+2-2g} \sum_{d_1=1}^{\left|\frac{1-d_L}{2}\right|} \lambda^{1-2d_1-d_L}([X]),$$

$$[\mathcal{M}_{\mathcal{L}}(3,d)] =$$

$$= \frac{\mathbb{L}^{-9d_L+9-9g}P_X(1)}{(\mathbb{L}-1)(\mathbb{L}^2-1)^2(\mathbb{L}^3-1)}\left(\mathbb{L}^{3g-1}(1+\mathbb{L}+\mathbb{L}^2)P_X(1)^2 - \mathbb{L}^{2g-1}(1+\mathbb{L})^2P_X(1)P_X(\mathbb{L}) + P_X(\mathbb{L})P_X(\mathbb{L}^2)\right)$$

$$+ \frac{\mathbb{L}^{-7d_L+5-5g}P_X(1)^2}{\mathbb{L}-1} \sum_{d_1=1}^{\left|\frac{1}{3}-\frac{d_L}{2}\right|} \left(\mathbb{L}^{d_1+g}\lambda^{-2d_1-d_L}([X]+\mathbb{L}^2) - \lambda^{-2d_1-d_L}([X]\mathbb{L}+1)\right)$$

$$+ \frac{\mathbb{L}^{-7d_L+5-5g}P_X(1)^2}{\mathbb{L}-1} \sum_{d_1=1}^{\left|\frac{2}{3}-\frac{d_L}{2}\right|} \left(\mathbb{L}^{d_1+g-1}\lambda^{-2d_1-d_L+1}([X]+\mathbb{L}^2) - \lambda^{-2d_1-d_L-1}([X]\mathbb{L}+1)\right)$$

$$+ \mathbb{L}^{-6d_L+3-3g}P_X(1) \sum_{d_1=1}^{-d_L} \sum_{d_2=\max\{d_L+d_1,1-d_1\}}^{\left|\frac{1-d_L-d_1}{2}\right|} \lambda^{-d_1+d_2-d_L}([X])\lambda^{1-d_1-2d_d-d_L}([X]),$$

where $\mathbb{L} = [\mathbb{C}]$. Analogous formulas are obtained for the corresponding E-polynomials.

Computational verification of Mozgovoy's conjecture

The previous formulas, combined with an algorithm capable of manipulating expressions in the λ-ring $\hat{K}_0(\mathcal{CM}_{\mathbb{C}})$, allowed us to **verify that Mozgovoy's conjectural formula [Moz12] for motives of moduli spaces of twisted Higgs bundles holds** when r and d are coprime and

$$2 \leq g \leq 11, \qquad 1 \leq r \leq 3, \qquad 2g-1 \leq -d_L \leq 2g+18.$$

The details of the computation and examples of the explicit polynomial expressions for the motives for low g, r and $-d_L$ can be accessed through the QR code on the right [Alf22].

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algorithm / computations

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