# FANO 3-FOLDS WITH 1-DIMENSIONAL MODULI

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## Introduction

K-stability detects the existence of Kähler-Einstein metrics on Fano Manifolds. Beyond that, it also provides the right platform for the construction of compact moduli of Fano varieties [6].
We are going to introduce the characterization of K-stability defined by [3, 5] which uses the stability threshold or $\delta$-invariant $\delta(X)$ of a Fano variety $X$.

## K-stability

Let $X$ be a Fano variety with kawamata log terminal singularities, let $f : \tilde{X} \to X$ be a birational morphism, such that $E$ prime divisor on , and let $n = \dim(X)$.
Define

$$S_X(E) = \frac{1}{(-K_X)^n} \int_0^\infty \mathrm{Vol}(f^*(-K_X) - tE)dt,$$

Then,

$$\delta(X) = \inf_E \frac{A_X(E)}{S_X(E)}$$

where $A_X(E) = 1 + \mathrm{ord}_E K_{\tilde{X}/X}$, and the infimum runs over all divisors $E$ over $X$.
We say $X$ is K-(semi)stable if $\delta(X) > 1$ ($\delta(X) \geq 1$).
Notice, that

K-stability $\Rightarrow$ K-polystability $\Rightarrow$ K-semistability.

## Fano 3-folds

There are 105 irreducible families of smooth Fano threefolds classified by Iskovskikh, Mori and Mukai. This project aims to explicitly describe K-moduli of smooth Fano threefolds for deformation families that depend on one parameter. There are 8 such families, but 3 do not contain any K-stable objects. For the remaining families, the K-stability of the smooth elements was studied in [2, 4]. Therefore, in order to compactify these K-moduli, we have to find special K-polystable limits for these families. Currently, this is done for all of them but one.

## Example: Twisted cubic and a line

One family is obtained by blowing up $\mathbb{P}^3$ along the disjoint union of a twisted cubic curve and a line. Let $X$ be smooth Fano threefold in this 1-parameter family. Then, $X$ can also be described as the blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ along a curve of degree $(3, 2)$. For an explicit description, let $S = \mathbb{P}^1 \times \mathbb{P}^1$, and let $C$ be a smooth curve in $S$ of degree $(1, 3)$. So in coordinates $([u : v], [x : y])$ on $S$ we descrive $C$ as:

1) $ux^3 + vy^3 = 0$,

2) $ux^2 + v(y^3 + y^2x) = 0$,

3) $u(x^3 + \lambda x^2y) + v(y^3 + \lambda y^2x) = 0$, where $\lambda \in \mathbb{C}^*$ such that $\lambda \neq \pm 1, \pm 3$.

These objects have many deformations. For instance, the twisted cubic can be deformed into 3 lines where 2 of them intersect the other one, or we could get a conic crossing a line, etc. But by [2] only one is K-polystable.

## Theorem

Let $L$, $L_1$, $L_2$, $L_3$ be lines in $\mathbb{P}^3$ given by the equations:

$$L : \begin{cases} x_0 = 0 \\ x_3 = 0 \end{cases}, \quad L_1 : \begin{cases} x_1 = x_0 \\ x_2 = 0 \end{cases}, \quad L_2 : \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}, \quad L_3 : \begin{cases} x_2 = x_3 \\ x_1 = 0 \end{cases}$$

and let $\pi : X \to \mathbb{P}^3$ be the blow-up of $\mathbb{P}^3$ along these lines. Then $X$ is K-polystable.

## Idea of the proof

(1) We assume, by contradiction that $X$ is not K-polystable. Then, by the definition, the $G$-equivariant small resolution $\hat{X}$ is also not K-polystable, i.e. there exist a divisor $F$ over $\hat{X}$ such that $\frac{A_{\hat{X}}(F)}{S(-K_{\hat{X}};F)} < 1$.

(2) Take a group acting on $\mathbb{P}^3$ that leaves invariant $L$, $L_2$ and $L_1 \cup L_3$. We choose $G = \mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}$, where

$$\mathbb{C}^* : \begin{cases} x_0 \to tx_0 \\ x_1 \to tx_1 \\ x_2 \to x_2 \\ x_3 \to x_3 \end{cases}, \text{ where } t \in \mathbb{C}^* \text{ and } \mathbb{Z}/2\mathbb{Z} : \begin{cases} x_0 \to x_3 \\ x_1 \to x_2 \\ x_2 \to x_1 \\ x_3 \to x_0 \end{cases}$$

By [7, Corollary 4.14] it is enough to study $G$-invariant divisors.

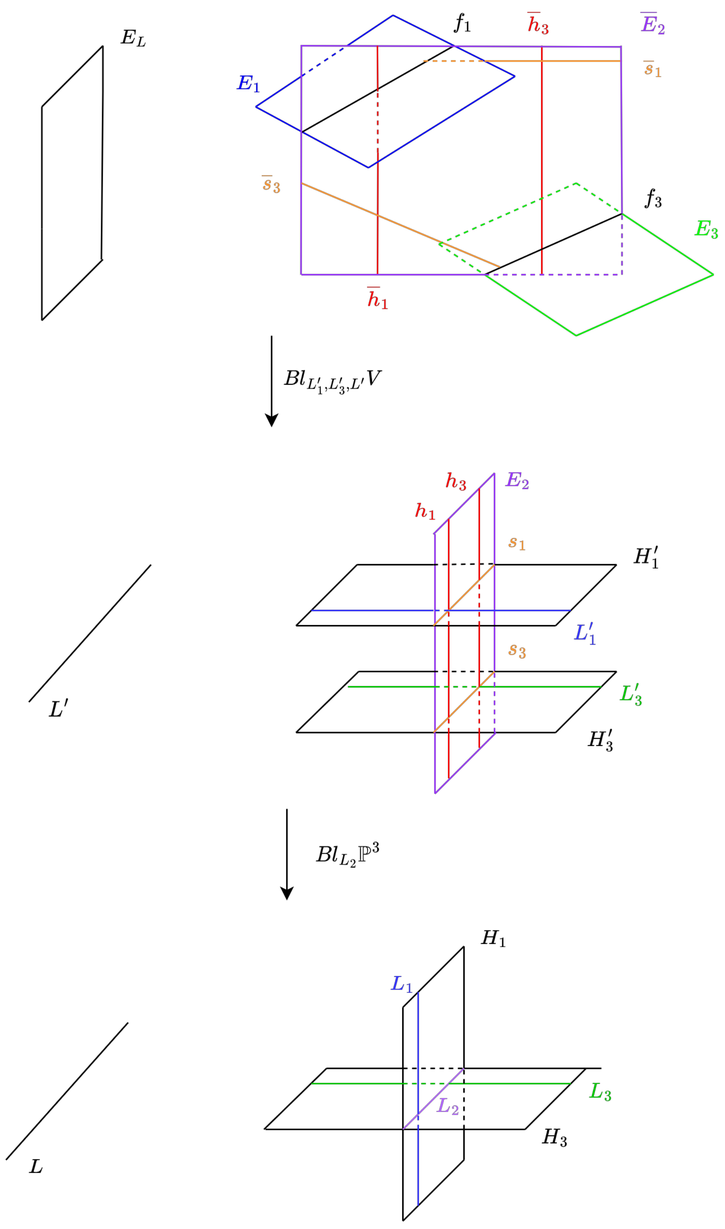



Fig. 1: Small resolution $\hat{X}$ of X.

(3) There are no $G$-invariant points in $\hat{X}$.

(4) We find a $G$-invariant admissible flag in $\hat{X}$ and use Abban-Zhuang theory [1] and [2, Corollary 1.7.26]. To get a lower bound for $\delta(\hat{X})$. And check that non of the possible flags give us a number lower than 1.
For example if we take the following flag: Divisor $\overline{E}_2$ in $\hat{X}$ and curve $g = f_1 + \overline{s}_1$ in $\overline{E}_2$. And we get $\frac{A_{\hat{X}}(E_2)}{S(-K_{\hat{X}};E_2)} = \frac{56}{51} > 1$.

(5) $\hat{X}$ is K-polystable. Therefore, our assumption is wrong and $X$ is K-polystable.

## Conclusion

With the previous theorem, we find the K-polystable element that will be the limit of the smooth K-polystable ones, compactifying the 1-dimensional moduli of that Fano family. With a similar process, we compactify the moduli of the remaining Fano families.

## References

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