# Geometric Aspects of the Swampland
## — Syllabus —

## Part 1

**Irene Valenzuela – Introduction to the Swampland Conjectures**

- Introduction to the Swampland program: motivation and goals.
- Brief description of the most relevant Swampland conjectures that will be relevant for this workshop, including the absence of global symmetries, the Distance conjecture and the Weak Gravity conjecture.
- Towards translating the Swampland conjectures into mathematical statements about the geometry and topology of the string compactification manifolds.

Refs. [1, 2].

**Arun Debray – Introduction to cobordism theory**

- Introduction to cobordism theory: What's a cobordism group, Thom's construction, spectra.
- How to compute cobordism: Adams and AHSS spectral sequences.
- Cobordism in condensed matter physics: topological superconductors, insulators, and all that.

Refs. Intro notes: [3]. Expository article on how to use the Adams spectral sequence to compute (twisted) spin bordism groups, including several examples: [4]. Classification of reflection positive invertible field theories in terms of bordism, discusses applications to SPT phases in condensed-matter physics, and more: [5].

**Miguel Montero – Cobordism theory and its connection to quantum gravity**

- Cobordism in condensed matter physics: topological superconductors, insulators, and all that.
- Cobordism in quantum field theory: anomalies in particle physics, connections to grand unified theories.
- Cobordism in quantum gravity: Cobordism conjecture, cobordism groups of string compactifications, discrete parameters in string/M theory.
- Open math problems: APS theorems for generalized spin structures.

Refs. [6–9].

**Andrzej Weber – The elliptic genus**

We will discuss the origins of the notion of elliptic genus. We wish to review the following subjects:

- Formal group laws and complex oriented cohomology theories.
- Hirzebruch construction of characteristic classes and genera.
- Elliptic formal group laws and elliptic genera.
- Rigidity property for Spin manifolds - Two parameter elliptic genus for complex manifolds associated to the Jacobi theta function.
- Elliptic genus as the localized $\chi_y$-genus of the free loop space.
- Transformation properties of elliptic genera.
- Mirror duality for Calabi-Yau hypersurfaces in toric varieties.
- Borisov-Libgober elliptic genus and elliptic class for singular varieties.

Refs. [10–14].

**Timo Weigand – The elliptic genus in four dimensions, quasi-Jacobi forms, and Gromov-Witten invariants on Calabi-Yau fourfolds**

Abstract: These lecture will introduce the concept of the elliptic genus of heterotic strings in four-dimensional compactifications with N=1 supersymmetry and discuss how its modular properties underlie a possible proof of the Weak Gravity Conjecture in certain regimes

of string theory. As a novelty in four-dimensional theories, compared to the familiar six-dimensional case, the elliptic genus transforms a quasi-Jacobi form, rather than a Jacobi, under the modular group of the torus. This is a consequence of an elliptic anomaly equation, which complements the familiar modular anomaly equation of the elliptic genus in six dimesions. We will sketch the derivation of this elliptic anomaly equation via topological string theory. The derivation makes use of string dualities and mirror symmetry on fourfolds, which result in an interpretation of the elliptic genus as a generating function for relative Gromov-Witten invariants on elliptically fibered Calabi-Yau fourfolds.

Refs. [15–17].

**Lecture 1:** The elliptic genus of heterotic strings and its realisation via F-theory on elliptic Calabi-Yau fourfolds:

I will discuss which relative Gromov-Witten invariants to compute in F-theory and how they appear in the elliptic genus of a dual heterotic theory

### Georg Oberdieck – Gromov-Witten theory of elliptic fibrations

**Lecture 2:** Gromov-Witten theory of elliptic fibrations and holomorphic anomaly equations

I will explain the mathematical formation of the holomorphic anomaly equation for the base-relative Gromov-Witten theory of elliptic fibrations. This is based on joint work with A. Pixton.

**Lecture 3:** Donaldson-Thomas theory of elliptic threefolds and holomorphic-anomaly equations

I will discuss holomorphic anomaly equations for the base-relative Donaldson-Thomas theory of elliptically fibered threefolds with section. We start with the work of Huang-Katz-Klemm on elliptic Calabi-Yau threefold and then consider the non-Calabi-Yau case.

Refs. [18].

### Timo Weigand (cont.)

**Lectures 2 and 3:** Applications of the holomorphic anomaly equations

- Application of holomorphic anomaly equation in GW theory on 4-folds to compute the elliptic genus of the heterotic string.
- Application to Weak Gravity Conjecture and Swampland Distance/Emergent String Conjecture.
- If time permits: Physics derivation of the holomorphic anomaly equation on elliptic 4-folds via BCOV.

# Part 2

## Thomas Grimm – Asymptotic Hodge theory

**Lecture 1:** Introduction

- Brief introduction to the swampland program with a focus on the Distance Conjecture and the finiteness of effective theories.
- Swampland conjectures in geometric string theory settings and their mathematical formulation. Some basics on Calabi-Yau threefolds/fourfolds, and their moduli spaces.

Refs. [20–22].

## Matt Kerr – Asymptotics of Variations of Hodge structure

**Lecture 1:** Variations of Hodge structure

The first lecture in this series will give an overview of polarized VHSs and period maps, including old and new results on local and global behavior. After a discussion of horizontality and special subvarieties, we discuss compactifications of period maps, which will motivate the subsequent lectures.

**Lecture 2:** Asymptotics in 1 variable

For a VHS over a punctured disk, the basic results are Borel's monodromy theorem and Schmid's nilpotent and $SL_2$-orbit theorems, which lead to the construction of the LMHS (limiting mixed Hodge structure). In the geometric case, the Clemens-Schmid sequence and vanishing-cycle sequence connect this to the mixed Hodge structure on the cohomology of the singular fiber and the Steenbrink spectrum of its singularities. Several examples will be given, including nodes on hypersurfaces, Kulikov degenerations of K3s and their generalization to Calabi-Yau varieties.

**Lecture 3:** Asymptotics in several variables

Concentrating on the 2-variable case, we describe constraints on asymptotics resulting from the Cattani-Kaplan-Schmid $SL_2$-orbit theorem together with Lie-theoretic considerations. We conclude with examples of the resulting typology for nilpotent cones for Calabi-Yau 3-folds, including a Hodge-theoretic compactification of an explicit 2-parameter period map.

Refs. [23–25].

## Benjamin Bakker – O-minimality and period maps

**Lecture 1:** Introduction to o-minimality

O-minimal structures provide a precise notion of tame asymptotics for real functions. Functions that are "definable" in an o-minimal structure have many of the same finiteness properties as real semialgebraic functions. In this talk we will give a hands-on introduction to o-minimal structures including lots of examples, with particular attention paid to period integrals in families of algebraic varieties.

**Lecture 2:** Definable structures on period spaces

A definable complex manifold is something between a complex algebraic manifold and a complex analytic manifold. It allows for non-algebraic functions locally but globally has the same finiteness properties as a compact algebraic manifold. In this talk we will carefully describe the definable complex manifold structure on spaces parametrizing periods, the definability of period maps with respect to this structure, and some important consequences.

**Lecture 3:** Period integrals in families of algebraic varieties

Period integrals—-namely, integrals of algebraic differential forms along topological cycles—-have been studied since the beginning of algebraic geometry and are at the heart of Hodge theory. In a family of algebraic varieties, such period integrals yield interesting holomorphic functions, which for instance solve a Picard—Fuchs differential equation. In this talk we'll revisit period integrals from the perspective of definable complex geometry, and describe some more advanced applications of o-minimal methods to the transcendence of these functions.

## Thomas Grimm (cont.)

**Lecture 2:** $Sl_2$ orbits and applications to Kähler cones

- Asymptotic form of the periods and nilpotent orbits
- Multi-variable $Sl_2$-orbit and the commuting Sl2s inducing a special split of the cohomology groups. Classification of asymptotic regions by enhancement diagrams
- Asymptotic behavior of the Hodge inner product

**Lecture 3:** Asymptotic behavior of the Hodge norm and its applications

- Application of asymptotic Hodge theory to give evidence for the Distance Conjecture
- Proof of the finiteness of F-theory vacua with self-dual fluxes using tame geometry/o-minimal structures and their manifestation in Hodge theory.

**Jacopo Stoppa – Introduction to CV ad tt* structures**

I will offer a brief introduction to certain special structures on vector bundles, such as CV structures on auxiliary bundles and tt* structures on tangent bundles [19]. These have important applications in enumerative geometry, also in the infinite-dimensional case [26], and play a role for the topic of this workshop [27].

# References


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