Fourier analysis with non commutative groups

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Plan

1. Fourier synthesis

2. CB Fourier multipliers

3. On the proofs : Hörmander–Mikhlin

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Fourier synthesis

Very classical theorem

If $1 < p < \infty$ and $f \in L_p(\mathbf{R}/\mathbf{Z})$. Define $S_N f(t) = \sum_{n=-N}^{N} \hat{f}(n) e^{2i\pi nt}$. Then

$$\lim_N \|f - S_N f\|_p = 0. \tag{1}$$

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Formally equivalent to the boundedness of the Hilbert transform :

- $HT \implies (1)$: $\|S_N\|_{p\to p} \leq \|H\|_{p\to p}^2$ + density of trigonometric polynomials.
- $(1) \implies HT$: by Banach–Steinhaus, $\sup_n \|S_N\|_{p\to p} < \infty$.

Fourier series : L^p convergence

The convergence $\lim_N \|f - S_N f\|_p = 0$ is false for $p = 1, \infty$: if

$$W_N(t) = \sum_{-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{2i\pi nt},$$

then $W_N \geq 0$ (so $\|W_N\|_1 = 1$), but $\|HW_N\|_1 \simeq \log n$.

Fourier synthesis in L^1 or L^∞

Theorem (Fejér)

If $1 \leq p \leq \infty$ and $f \in L_p(\mathbf{R}/\mathbf{Z})$ (with f continuous if $p = \infty$), then

$$\lim_N \|f - W_N * f\|_p = 0.$$

$W_N * f(t) = \sum_{-N}^{N} \left(1 - \frac{|n|}{N}\right) \hat{f}(n) e^{2i\pi nt}$.

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More generally, for every (countable) abelian group Γ, consider its Pontryagin dual $\hat{\Gamma}$ (group of homomorphisms $\Gamma \to \{z \in \mathbf{C} : |z| = 1\}$). This is a compact group, with Haar probability measure.

There is a sequence of $\varphi_N : \Gamma \to \mathbf{C}$ finitely supported such that, for every $f \in L_p(\hat{\Gamma})$ (with $f \in C^0$ if $p = \infty$),

$$\|f - \sum_\gamma \varphi_N(\gamma)\hat{f}(\gamma)\gamma\|_p = 0.$$

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Here we replace Γ. But then what is $L_p(\hat{\Gamma})$? We need to define a norm on the vector space with basis Γ (the trigonometric polynomials), $\mathbf{C}[\Gamma]$. Then define $L_p(\hat{\Gamma})$ as the completion of $\mathbf{C}[\Gamma]$ for this norm.

The norm in $L_p(\hat{\Gamma})$

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$$\|a\|_p^p := \sum_{\gamma_1,\dots,\gamma_{2m},\gamma_1\gamma_2\dots\gamma_{2m}=1} a_{\gamma_1}\overline{a_{\gamma_2^{-1}}}\dots\overline{a_{\gamma_{2m}^{-1}}}.$$

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Operator algebraic definition : if $\lambda(a) \in B(\ell_2(\Gamma))$ is the operator by left-convolution by a : $\lambda(a)f(\gamma_0) = \sum_\gamma a_\gamma f(\gamma^{-1}\gamma)$, then define

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Fact : these two formulas define the same quantity, which is a norm (the L_p space of the von Neumann algebra of Γ).

It satisfies $(L_p)^* = L_{p'}$ and Hölder $|\sum_\gamma a_\gamma b_\gamma| \leq \|a\|_p\|b\|_{p'}$.

Fourier synthesis

Every $f \in L_p(\hat{\Gamma})$ has Fourier coefficients f "$=$" $\sum_\gamma \hat{f}(\gamma)\gamma$".

We say that Γ satisfies L_p-**Fourier synthesis** if there is a sequence of finitely supported functions $\varphi_N : \Gamma \to \mathbf{C}$ such that

$$\lim_N \|f - \sum_\gamma \varphi_N(\gamma)\hat{f}(\gamma)\gamma\|_p = 0.$$

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Conjecture

$\mathrm{SL}_{d\geq 3}(\mathbf{Z})$ does not satisfy L_p-Fourier synthesis for any $p > 4$.

Equivalently for $p < \frac{4}{3}$. Perhaps even $p \neq 2$?

For $p = \infty$ or $p = 1$ (weakest form), this is a consequence of the (conjectured) fact that the Banach space $C^*_{red}(\mathrm{SL}_{d\geq 3}(\mathbf{Z}))$ does not have **Grothendieck's approximation property**.

CB Fourier multipliers

A technical but important variant : operator coefficients

Let S_p denote the Schatten p-class : the space of $T \in B(\ell_2)$ such that $Tr(|T|^p) < \infty$.

If $a = \sum_\gamma a_\gamma \gamma$ with $a_\gamma \in S_p$, we can make sense of the norm $\|a\|_p$. For example, if $p = 2m$,

$$\|a\|_p^p := \sum_{\gamma_1,\dots,\gamma_{2m},\gamma_1\gamma_2\dots\gamma_{2m}=1} Tr(a_{\gamma_1}\overline{a_{\gamma_2^{-1}}}\dots\overline{a_{\gamma_{2m}^{-1}}}).$$

The completion is denoted $L_p(\hat{\Gamma}; S_p)$.

Notation (Fourier multiplier) : If $\varphi : \Gamma \to \mathbf{C}$, write $T_\varphi : f \mapsto \sum_\gamma \varphi(\gamma)\hat{f}(\gamma)\gamma$, $\|T_\varphi\|_{p-cb}$ its norm on $L_p(\hat{\Gamma}; S_p)$ the completely bounded norm of the Fourier multiplier T_φ.

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In the same way as for Fourier transform vs Fourier series, all this makes sense for locally compact groups.

Theorem (Lafforgue–dlS 11, de Laat–dlS 16)

Let $\Gamma = \mathrm{SL}_3(\mathbf{Z})$. For every $4 < p < \infty$ or $1 \leq p < \frac{4}{3}$, there is $f \in L_p(\hat{\Gamma}; S_p)$ such that, for every finitely supported $W : \Gamma \to \mathbf{C}$,

$$\|f - \sum_{\gamma} W(\gamma)\hat{f}(\gamma)\lambda(\gamma)\|_p \geq 1.$$

If $\Gamma = \mathrm{SL}_{d\geq 3}(\mathbf{Z})$, the same holds for $|\frac{1}{p} - \frac{1}{2}| > \frac{c}{d-2}$.

Fundamental inequalities for $G = \mathrm{SL}_d(\mathbf{R})$:

There is $\varepsilon > 0$ such that if $\varphi\colon G \to \mathbf{C}$ is $\mathrm{SO}(d)$-biinvariant and a completely bounded L_p Fourier multiplier, then φ has a limit at infinity and

$$|\varphi(g) - \varphi_\infty| \lesssim \|T_\varphi\|_{p-cb}\|g\|^{-\varepsilon}.$$

Open question : what about for $|\frac{1}{p} - \frac{1}{2}| \leq \frac{c}{d-2}$?

Results with Javier Parcet and Éric Ricard

Denote $G = \mathrm{SL}_d(\mathbf{R})$ and $K = \mathrm{SO}(d)$.

Local Hörmander–Mikhlin theorem (PRdlS 22)

Let $\varphi\colon G \to \mathbf{R}$ be a compactly supported function with $\frac{d^2}{2} + 1$ derivatives such that $\sup_g d(g, 1)^k |D^k \varphi(x)| < \infty$. Then T_φ is completely bounded on $L_p(\widehat{G})$ for every $1 < p < \infty$.

Results with Javier Parcet and Éric Ricard

On the necessity of the hypotheses. $((G, K) = (\mathrm{SL}_d(\mathbf{R}), \mathrm{SO}(d))$.

Theorem (PRdlS 22)

Let $p > 2$ and $d \geq 3$ such that $\alpha := \frac{d-2}{2} - \frac{d-1}{p} > 0$. If $\varphi \colon (1, \infty) \to \mathbf{C}$ is a function such that $m(g) = \varphi(\|g\|)$ is a multiplier on $L_p(VNG)$, then φ is of class $C^{\alpha-}$. Moreover, there are C and $c_k > 0$ such that

- φ has a limit φ_∞ such that
$$|\varphi(x) - \varphi_\infty| \leq \|T_m\|_{cb} \frac{C}{x^{c_0}}$$
- For every integer $1 \leq k < \alpha$ and $x \in (1, \infty)$
$$|\varphi^{(k)}(x)| \leq \|T_m\|_{cb} \frac{C}{(x-1)^k x^{c_k}}.$$

Results with Javier Parcet and Éric Ricard

Analogous results in rank 1.

Let $G = \mathrm{SO}(d, 1)$ ($\simeq$ isometry group of real hyperbolic space of dimension $d + 1$). $K = O(d)$ maximal compact subgroup.

Theorem (PRdlS 22)

Same d, p, α. Then K-biinvariant cb multipliers of $L_p(\hat{G})$ are of class $C^{\alpha-}$, and similar estimates **without the c_k's**.

In particular,

Theorem (PRdlS 22)

Every K-finite matrix coefficient of **every** representation of $\mathrm{SO}(d, 1)$ on a Hilbert space is of class $C^{\frac{d}{2}-1-}$.

On the proofs : Hörmander–Mikhlin

Reminder of the statement

$G = \mathrm{SL}_d(\mathbf{R})$, $K = \mathrm{SO}(d)$.

[PRdlS] Local Hörmander–Mikhlin theorem

Let $m\colon G \to \mathbf{R}$ be a compactly supported function with $\frac{d^2}{2} + 1$ derivatives such that $\sup_g d(g, 1)^k |D^k m(x)| < \infty$. Then T_m is completely bounded on $L_p(\widehat{G})$ for every $1 < p < \infty$.

Fourier and Schur multipliers

If $m\colon G \to \mathbf{C}$ is a continuous function. The associated Herz-Schur multiplier is the map $S_m : (a_{g,h})_{g,h\in G} \mapsto (m(gh^{-1})a_{g,h})_{g,h\in G}$.

Classical Bozejko-Fendler theorem : for $p = \infty$, $\|T_m\|_{CB(VN(G))} = \|S_m\|_{CB(B(L_2G))}$.

For $1 < p < \infty$,

$$\|S_m\|_{CB(S_p(L_2G))} \leq \|T_m\|_{CB(L_p(\widehat{G}))},$$

with equality if G amenable (Neuwirth-Ricard, Caspers-dlS).

Example ($G = \mathbf{Z}$) triangular truncation and Hilbert transform.

Fourier and Schur multipliers, 2

Non-amenable G : wide open! But

Theorem (PRdlS 22)

Let $1 < p < \infty$. If $\Omega \subset G$ is compact, then there is a compact subset $\Sigma = \Sigma(\Omega, p) \subset G$ and $C = C(\Omega, p)$ such that

$$\|T_m\|_{CB(L_p(\hat{G}))} \leq C\|S_m\|_{CB(S_p(L_2\Sigma))}.$$

So the Theorem follows from

Theorem (Conde-Alonso–González-Pérez–Parcet–Tablate 22+)

Let Σ be a compact manifold of dimension d, and $\varphi \colon \Sigma \times \Sigma \to \mathbf{C}$ be a function of class $C^{\frac{d}{2}+1}$ outside of the diagonal, with a controlled explosion on the diagonal. Then the map $(a_{x,y}) \mapsto (m(x,y)a_{x,y})_{x,y}$ is bounded on $S_p(L_2\Sigma)$.

On the proofs : Harmonic analysis on the sphere and Lafforgue's rank 0 reduction

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Theorem [PRdlS]

Let $p > 2$ and $d \geq 3$ such that $\alpha := \frac{d-2}{2} - \frac{d-1}{p} > 0$. If $\varphi\colon (1,\infty) \to \mathbf{C}$ is a function such that $m(g) = \varphi(\|g\|)$ is a multiplier on $L_p(VNG)$, then φ is of class $C^{\alpha-}$. Moreover, there are C and $c_k > 0$ such that

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- For every integer $1 \leq k < \alpha$ and $x \in (1,\infty)$

$$|\varphi^{(k)}(x)| \leq \|T_m\|_{cb} \frac{C}{(x-1)^k x^{c_k}}.$$

Proof : Rank 0 reduction.

$$\mathbf{S}^{d-1} = \{x = (x_1, \ldots, x_d) \in \mathbf{R}^d \mid \sum x_i^2 = 1\}.$$

If $m : [-1, 1] \to \mathbf{C}$ is a function, we are interested in the Schur multiplier $S_m : B(L_2(\mathbf{S}^{d-1})) \to B(L_2(\mathbf{S}^{d-1}))$ given by $A = (a_{x,y})_{x,y \in \mathbf{S}^{d-1}} \mapsto (m(\langle x, y\rangle)a_{x,y})_{x,y \in \mathbf{S}^{d-1}}$.

Proposition (Lafforgue-dlS, de Laat-dlS, PRdlS)

Let $p > 2 + \frac{2}{d-2}$, et $\alpha := \frac{d-2}{2} - \frac{d-1}{p} > 0$. If S_m is bounded on $S_p(L_2(\mathbf{S}^{d-1}))$, then m is of class $C^{\alpha-}$ on $(-1, 1)$.

Harmonic analysis on the sphere $\mathbf{S}^{n-1}$

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Open question : what about when $2 < p < \frac{2}{d-2}$?

Toy question : (spherical Hilbert transform) Let $m(\delta) = 1_{\delta>0}$. Is S_m bounded on S_p for some $p \neq 2$?

Harmonic analysis on the sphere, 2

For $\delta \in [-1, 1]$, define an operator T_δ on $L_2(\mathbf{S}^{n-1})$

$$T_\delta f(x) = \text{average of } f \text{ on } \{y \in \mathbf{S}^{n-1} \mid \langle x, y\rangle = \delta\}.$$

(Lafforgue-dlS, de Laat-dlS) If $\delta \in (-1, 1)$, $T_\delta \in S_p((L_2(\mathbf{S}^{n-1})))$ if and only if $p > 2 + \frac{2}{n-2}$.

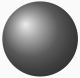
Let $\alpha := \frac{n-2}{2} - \frac{n-1}{p} > 0$.

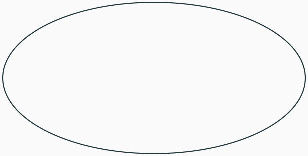
(Parcet-Ricard-dlS) The map $\delta \in (-1, 1) \mapsto T_\delta \in S_p(L_2\mathrm{SO}(n))$ if of class $C^{\alpha-}$ (C^α if $\alpha \notin \mathbf{N}$).

Weyl chamber exploration, $n = 3$ (Lafforgue)

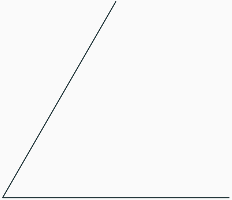
Take $G = \mathrm{SL}_3(\mathbf{R})$,
$K = \mathrm{SO}_3$ maximal compact.

$$U = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\} \cap K \simeq \mathrm{SO}(2).$$

K/U 

 G/K

$U \backslash K / U$ 

 $K \backslash G / K$

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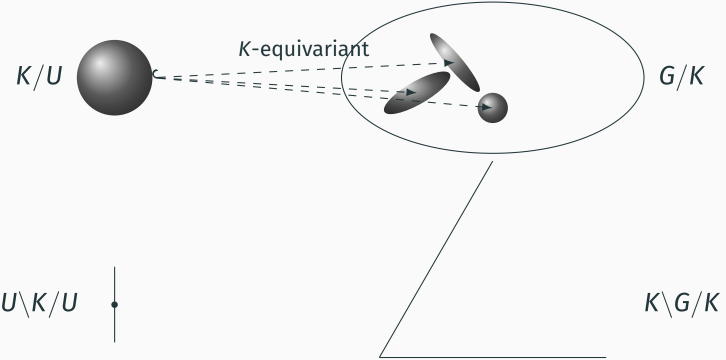

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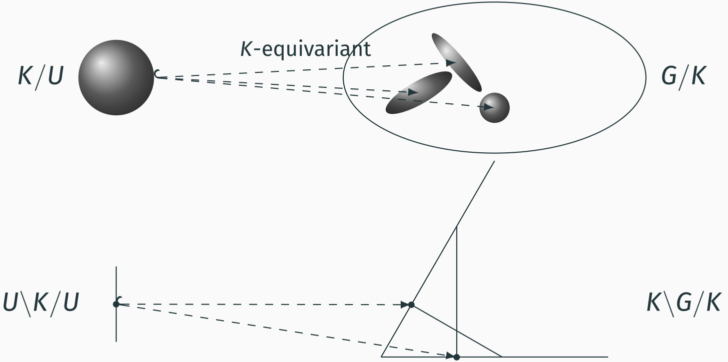

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