5TH INTERNATIONAL CONFERENCE ON HARMONIC ANALYSIS AND PARTIAL DIFFERENTIAL EQUATIONS

El Escorial (Spain) 17-21 June, 1996

(Dedicated to Professor Miguel de Guzmán)

COURSES

Tadeusz Iwaniec *(Syracuse University)*
Nonlinear commutators and current advances in PDE's

Pertti Mattila *(Universita Jyväskylä)*
Singular integrals, analytic capacity, and rectifiability

Fulvio Ricci *(Politecnico di Torino)*
Solvability of differential operators on the Heisenberg group

Peter Sjögren *(Chalmers Tekniska Högskola & Göteborg Universitet)*
Gaussian measure, associated Laplacian and operators

SPEAKERS

William Beckner (University of Texas at Austin)
Alexandra Bellow (Northwestern University)
Anthony Carbery (University of Edinburgh)
Michael Christ (University of California, Los Angeles)
Ronald Coifman (Yale University)
Antonio Córdoba (Universidad Autónoma de Madrid)
Eugene Fabes (University of Minnesota)
Peter Jones (Yale University)
Jean Pierre Kahane (Université Paris-Sud)
Carlos Kenig (University of Chicago)
Yves Meyer (Université Paris-Dauphine, CEREMADE)
Richard Rochberg (Washington U. in St. Louis)
Yoram Sagher (University of Illinois at Chicago Circle)
Carlos Segovia (U. de Buenos Aires/ U.A.M.)
Fernando Soria (Universidad Autónoma de Madrid)
Mitchell Taibleson (Washington U. in St. Louis)
Guido Weiss (Washington U. in St. Louis)
Grant Welland (University of Missouri at St. Louis)

ORGANIZING COMMITTEE

Patricio Cifuentes, José García-Cuerva, Eugenio Hernández, Fernando Soria, José Luis Torrea
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E-mail: escorial@ccuam3.sdi.uam.es

UAM UNIVERSIDAD AUTONOMA DE MADRID

6TH INTERNATIONAL CONFERENCE ON HARMONIC ANALYSIS AND PARTIAL DIFFERENTIAL EQUATIONS

El Escorial (Spain) 3-7 July, 2000

Short Courses

- **Prof. G. Mauceri** (U. Genova) Functional calculus for the Ornstein-Uhlenbeck semigroup.
- **Prof. A. Seeger** (U. Wisconsin, Madison) Fourier integral operators with degeneracies.
- **Prof. C. Thiele** (U. California, Los Angeles) Multilinear singular integrals.
- **Prof. A. Volberg** (Michigan State U., East Lansing) Nonhomogeneous Harmonic Analysis and applications.

Invited Speakers

B. Beckner (Austin, TX)
A. Carbery (Edinburgh)
M. J. Carro (Barcelona)
J. Duoandikoetxea (Bilbao)
A. Gillespie (Edinburgh)
S. Hofmann (Columbia, MO)
P. Jones (New Haven, CT)
P. Mattila (Jyväskylä)
P. MacManus (Maynooth)
Y. Meyer (Paris)
J. Pipher (Providence, RI)
F. Ricci (Torino)
P. Sjögren (Göteborg)
T. Tao (Los Angeles, CA)
R. Torres (Lawrence, KS)
J. Verdera (Barcelona)
S. Wainger (Madison)
G. Weiss (St. Louis, MO)

ORGANIZING COMMITTEE

Patricio Cifuentes, José García-Cuerva, Eugenio Hernández,
Fernando Soria, José Luis Torrea, Ana Vargas
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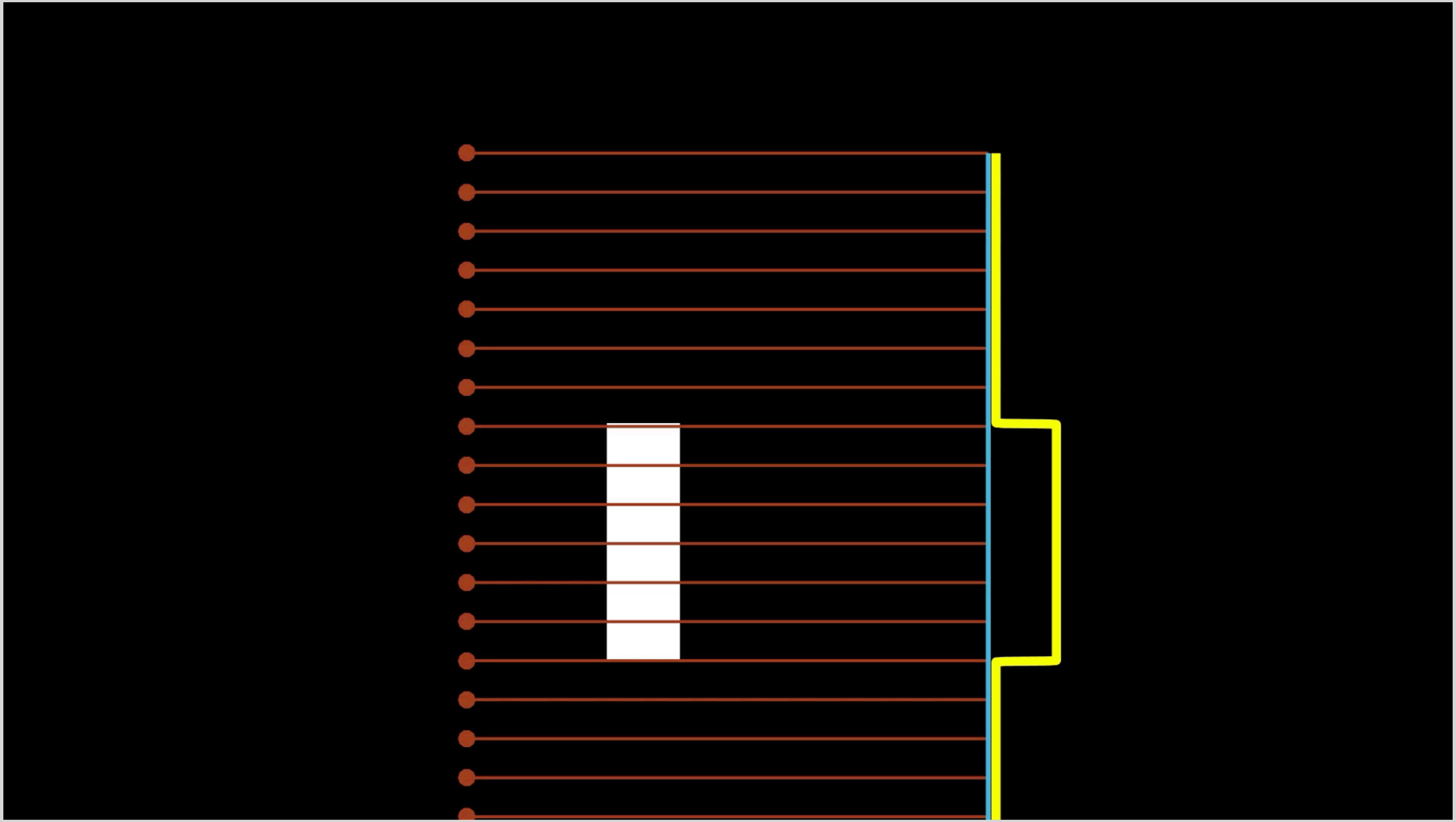
3ecm BARCELONA 2000 — D.G.E.S. — UAM UNIVERSIDAD AUTONOMA DE MADRID — 2000 WORLD MATHEMATICAL YEAR 2000

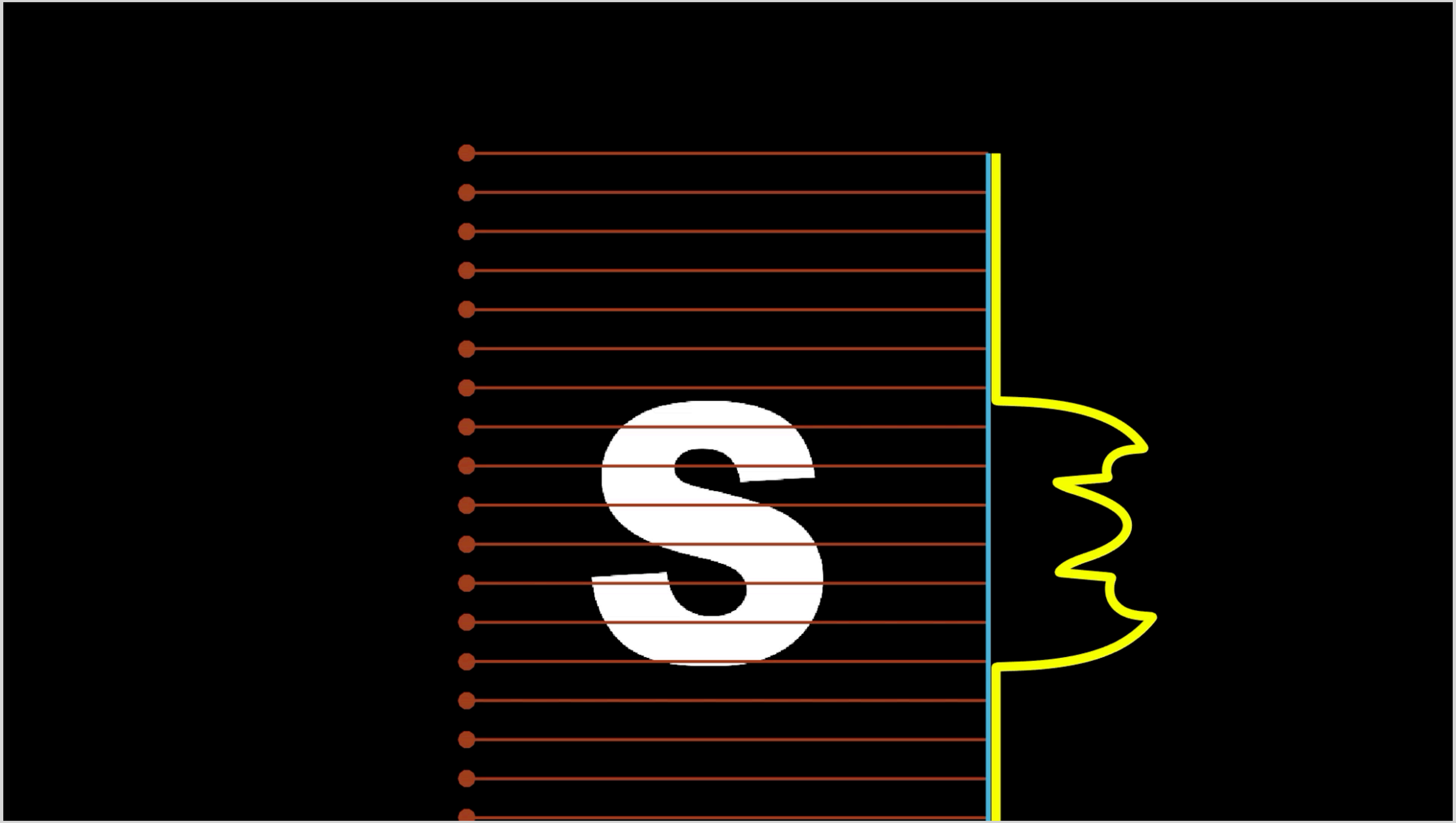
Unitarization of the Radon Transform

11th International Conference on Harmonic Analysis and Partial Differential Equations
June 6-10, 2022, El Escorial (Madrid)

Filippo De Mari
joint work with G. Alberti, F. Bartolucci, E. De Vito, M. Monti and F. Odone



many thanks to Samuli Siltanen



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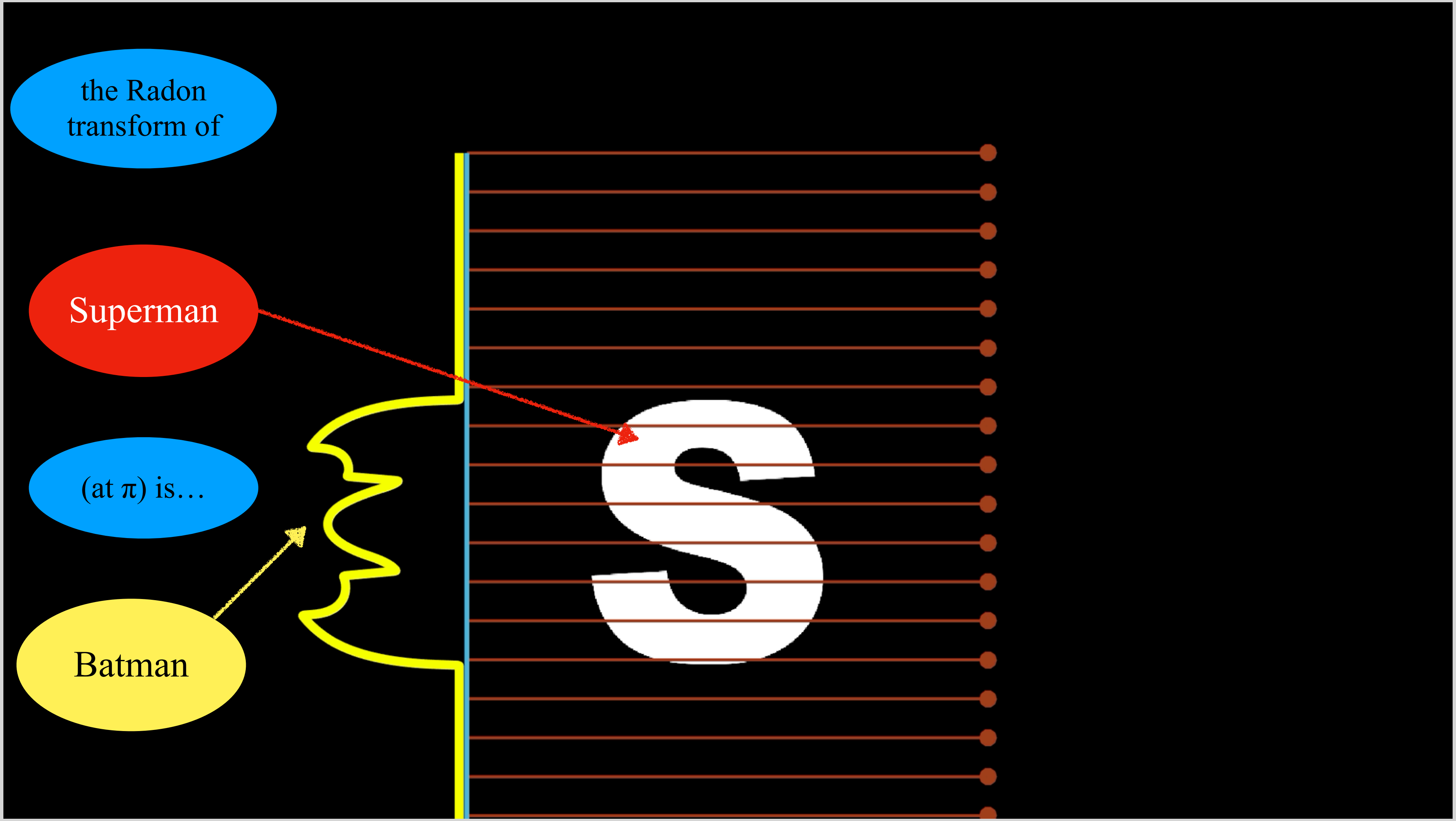

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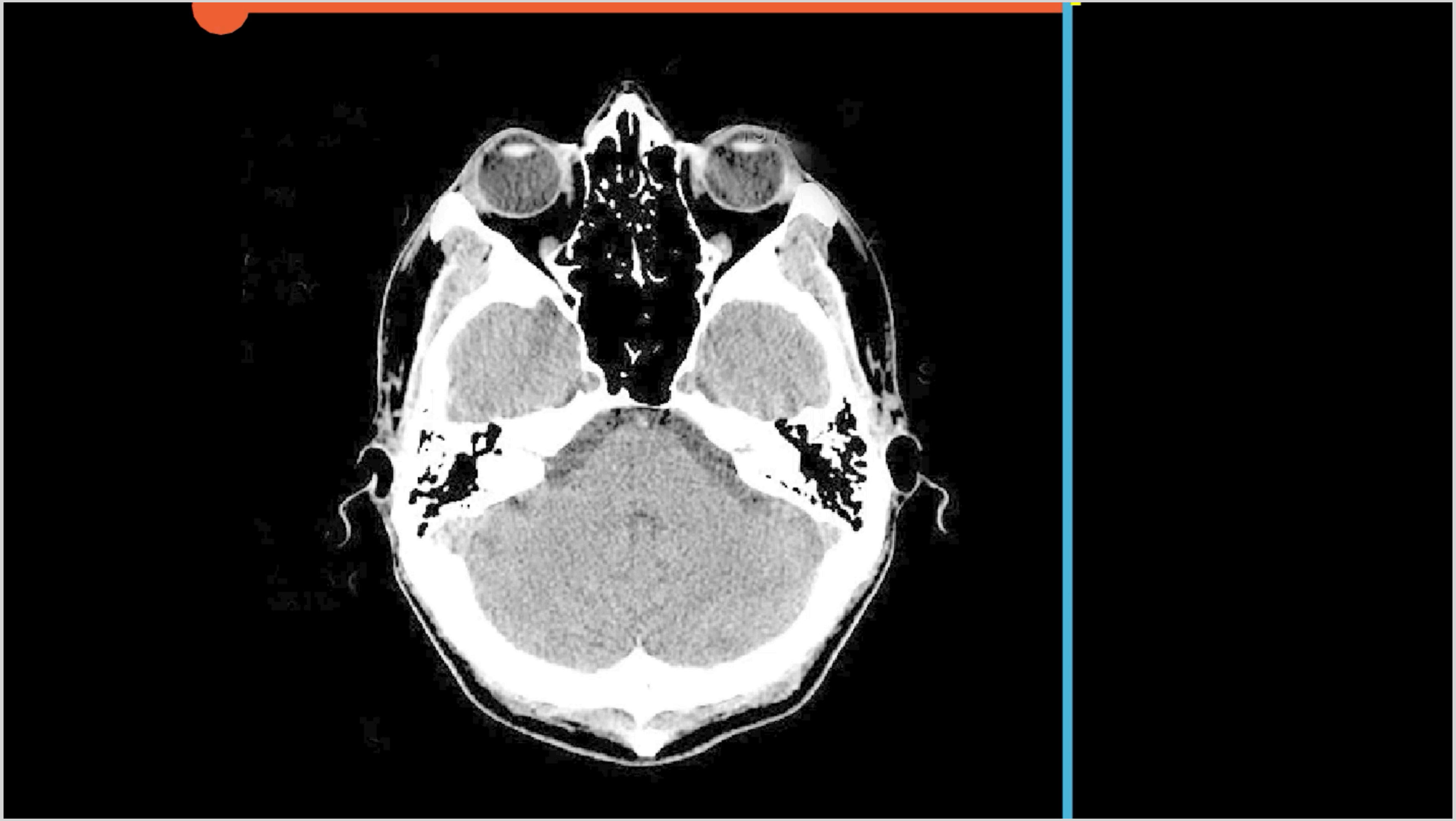

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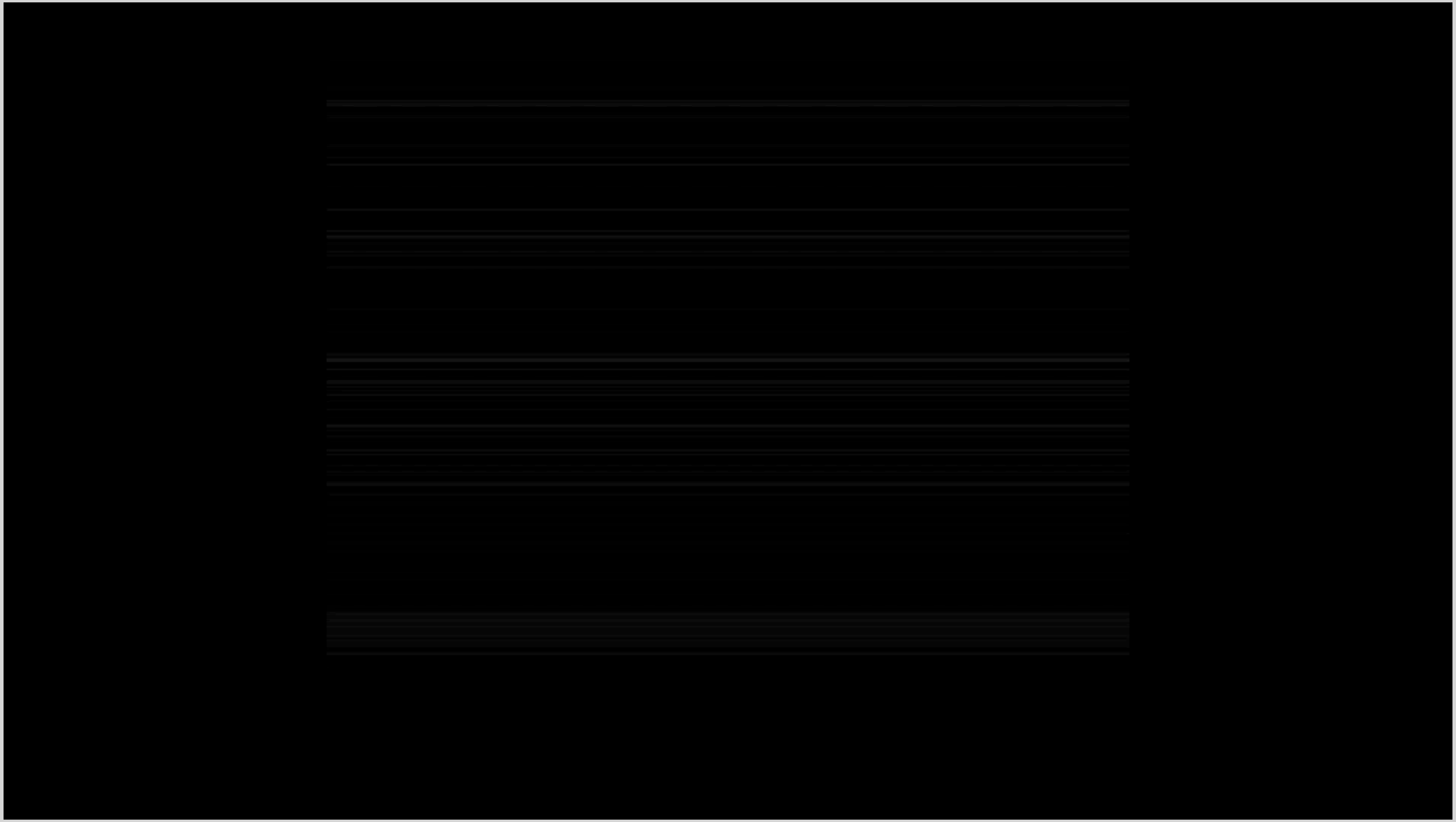

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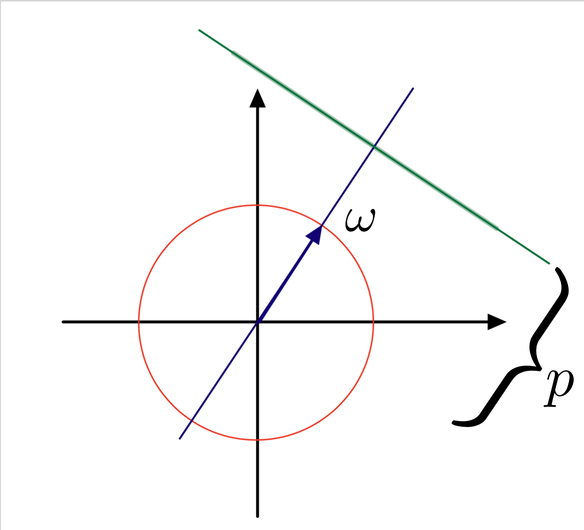
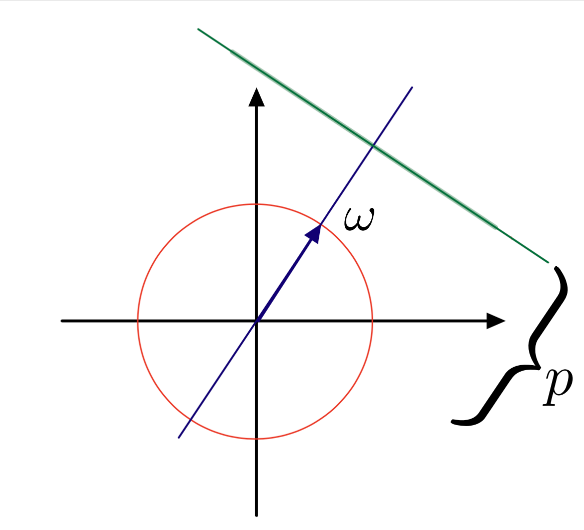
the Radon transform



$$\mathcal{R}f(\omega, p) = \int_{y\cdot\omega=p} f(y)\mathrm{d}y$$

the Radon transform



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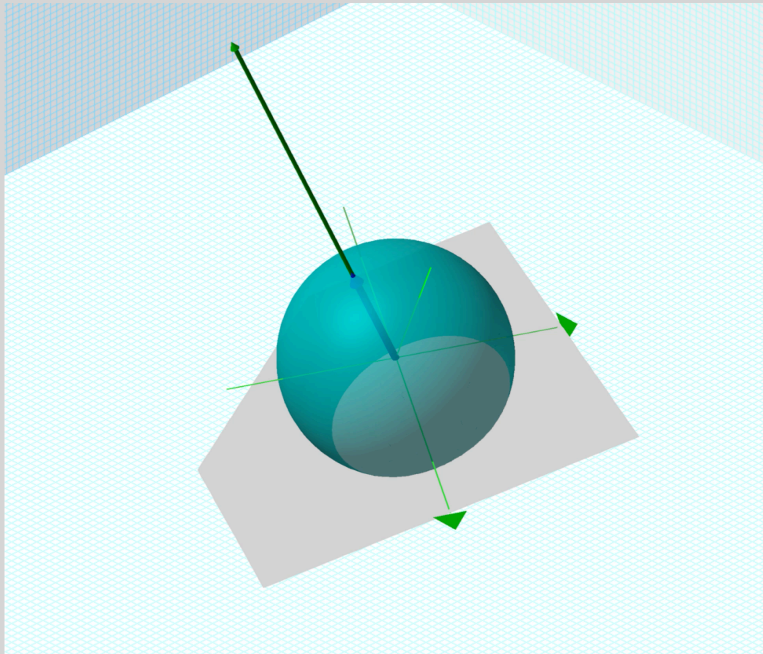
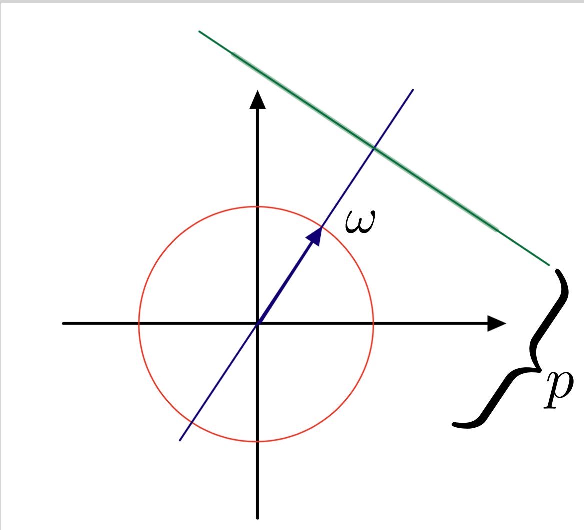


$$(\omega,p) \in S^{d-1} \times \mathbb{R}/\sim$$

inversion and unitarization

$$\mathcal{R}f(\omega,p) = \int_{y\cdot\omega=p} f(y)\mathrm{d}y$$

$$\mathcal{R}^{\#}F(x) = \int_{S^{d-1}} F(\omega, x\cdot\omega)\mathrm{d}\sigma(\omega)$$



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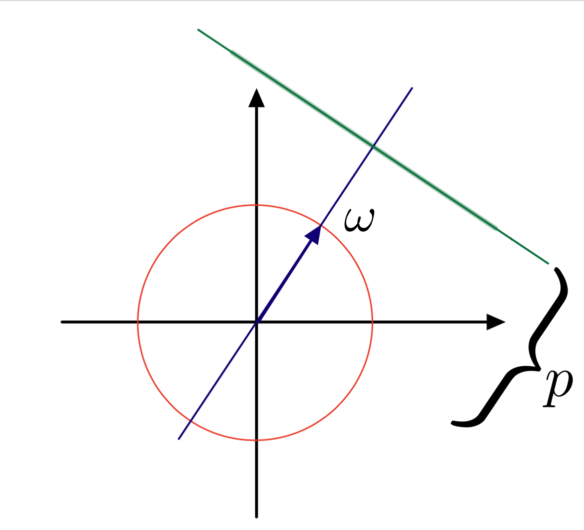

- Riesz potentials: if $\alpha < d$, put $\big(\mathbf{I}^{\alpha} f\big)^{\wedge}(\xi) = |\xi|^{-\alpha}\widehat{f}(\xi)$

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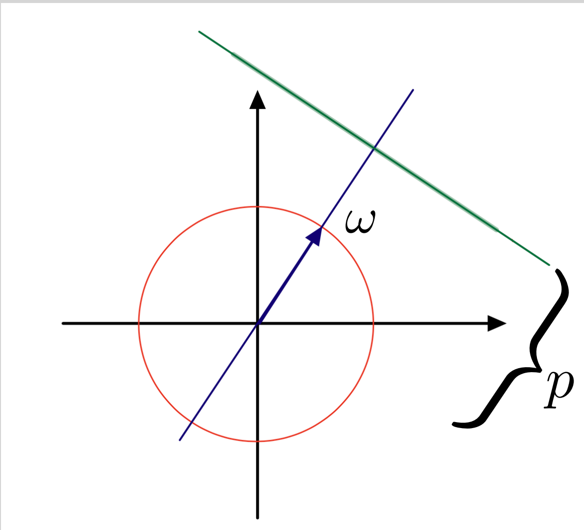

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$$f = \frac{1}{2}\mathbf{I}^{-\alpha}\mathcal{R}^{\#}\mathbf{I}^{\alpha-d+1}\mathcal{R}f$$

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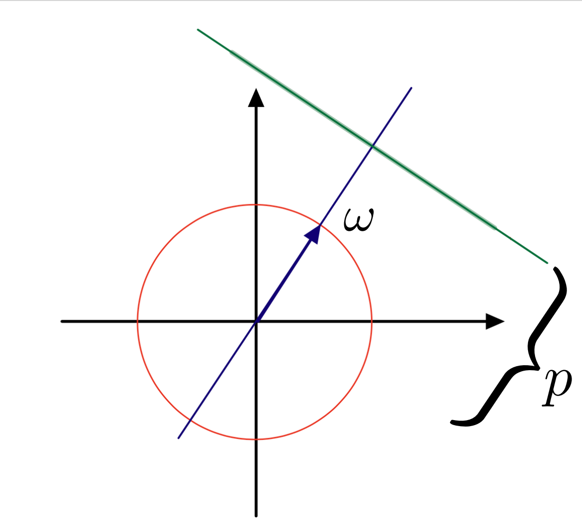

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- $\alpha = \dot{\alpha} = \dfrac{d-1}{2} < d \implies -\dot{\alpha} = \dot{\alpha} - d + 1$

inversion and unitarization

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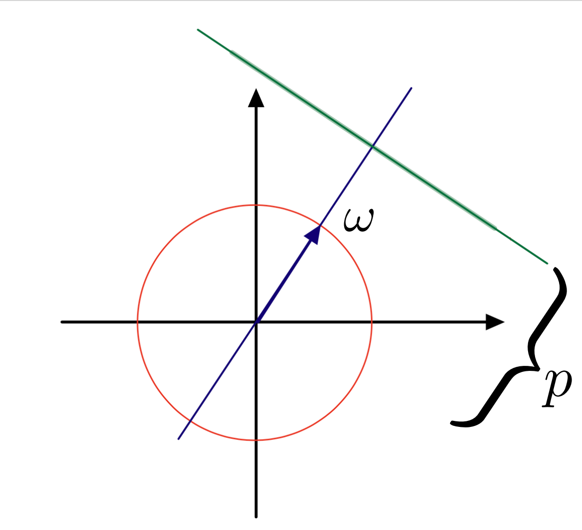

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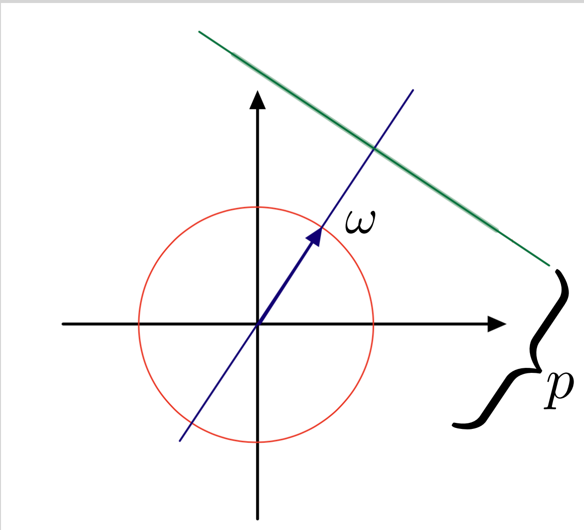

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Theorem (S. Helgason) The map $f \mapsto \mathbf{I}^{-\dot{\alpha}}\mathcal{R}f$ extends to a unitary operator $\mathcal{Q}$: $L^2(\mathbb{R}^d) \to L^2_{\text{even}}(S^{d-1}\times\mathbb{R})$.

inversion and unitarization

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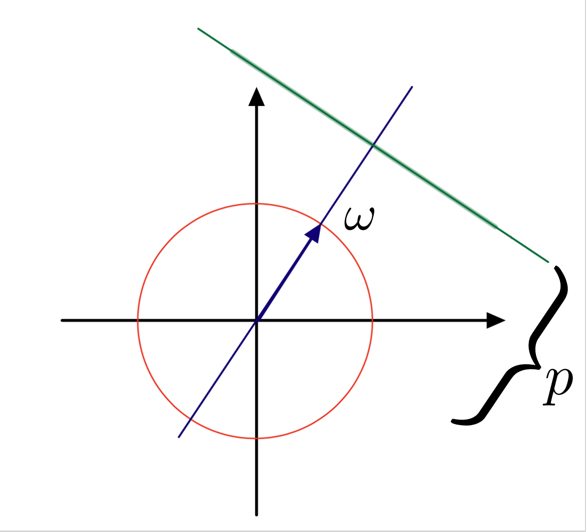

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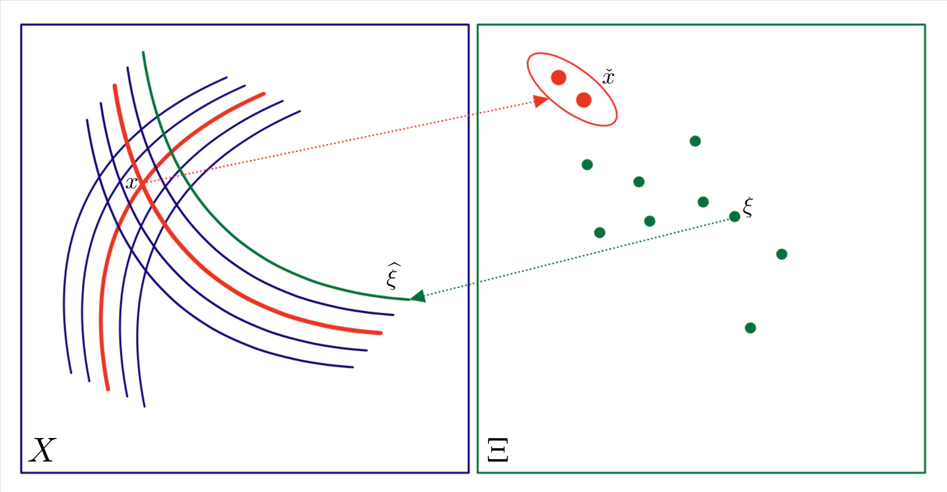
- for $d=2$, we have $\widehat{(\mathbf{I}^{-\dot{\alpha}})}(\xi) = |\xi|^{1/2}\widehat{f}(\xi)$

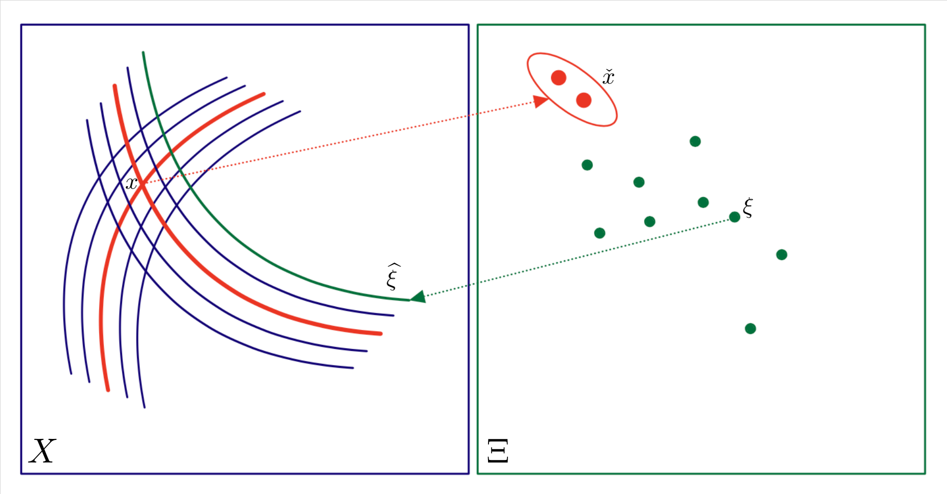
- for $d=3$, we have $\widehat{(\mathbf{I}^{-\dot{\alpha}})}(\xi) = |\xi|\widehat{f}(\xi)$

geometric duality and Radon transform



geometric duality and Radon transform

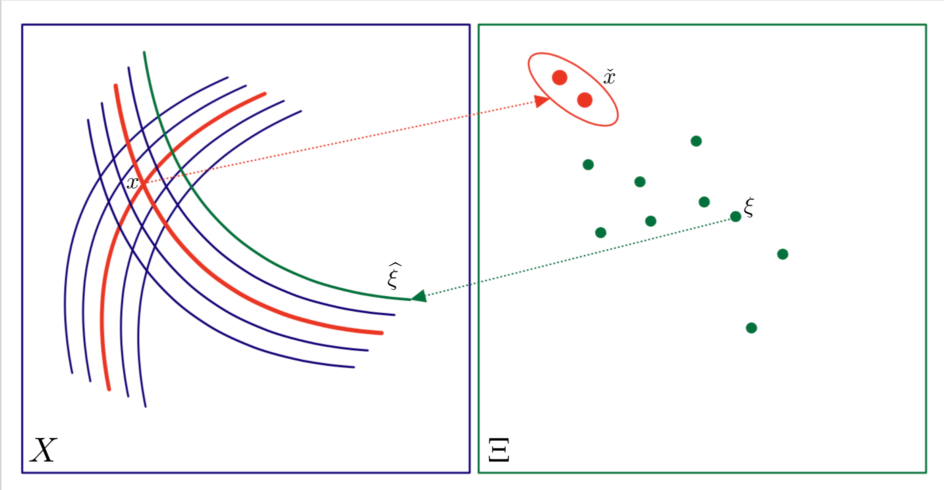
“input” space



“label” space

geometric duality and Radon transform

“input” space

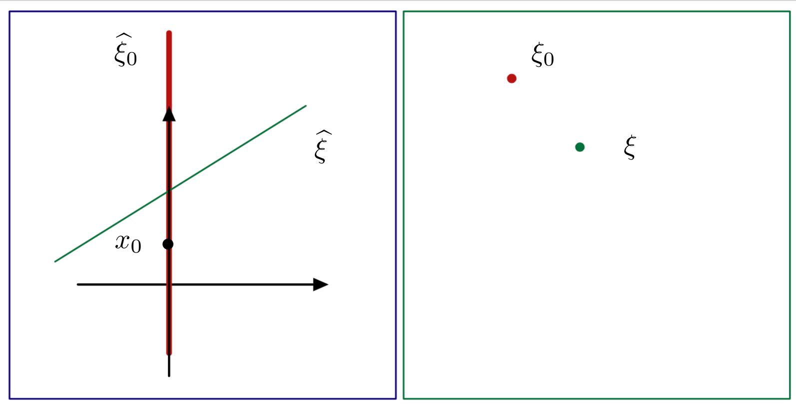


“label” space

Definition. Given the family $\{\hat{\xi}\}_{\xi\in\Xi}$ of subsets of X, each endowed with a measure $\mathrm{d}m_\xi$, the Radon transform of $f\colon X \to \mathbb{C}$ is the function $\mathcal{R}f\colon \Xi \to \mathbb{C}$ given by:

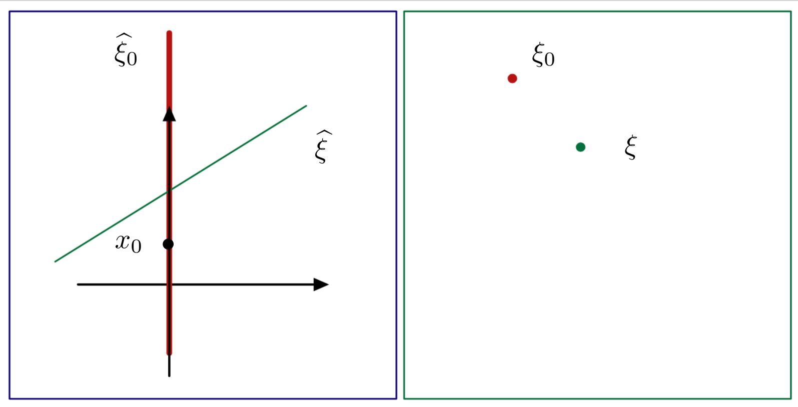
$$\mathcal{R}f(\xi) = \int_{\hat{\xi}} f(x)\mathrm{d}m_\xi(x)$$

dual pairs - homogeneous spaces



$X \simeq G/K$ $\Xi \simeq G/H$

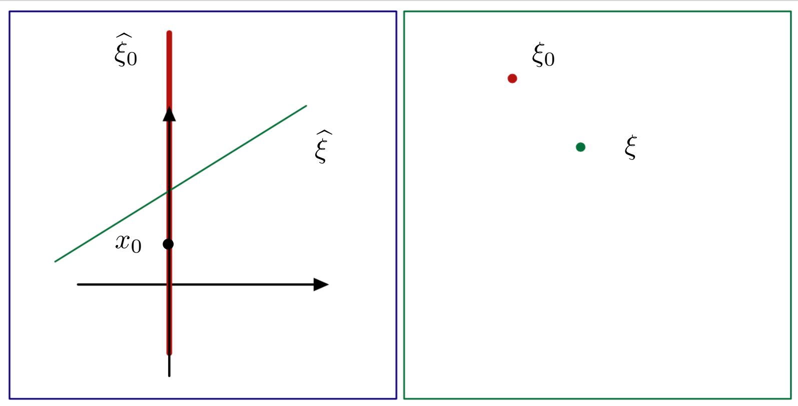
dual pairs - homogeneous spaces



$X \simeq G/K$ $\Xi \simeq G/H$

- G a lcsc topological group
- X, Ξ transitive G-spaces

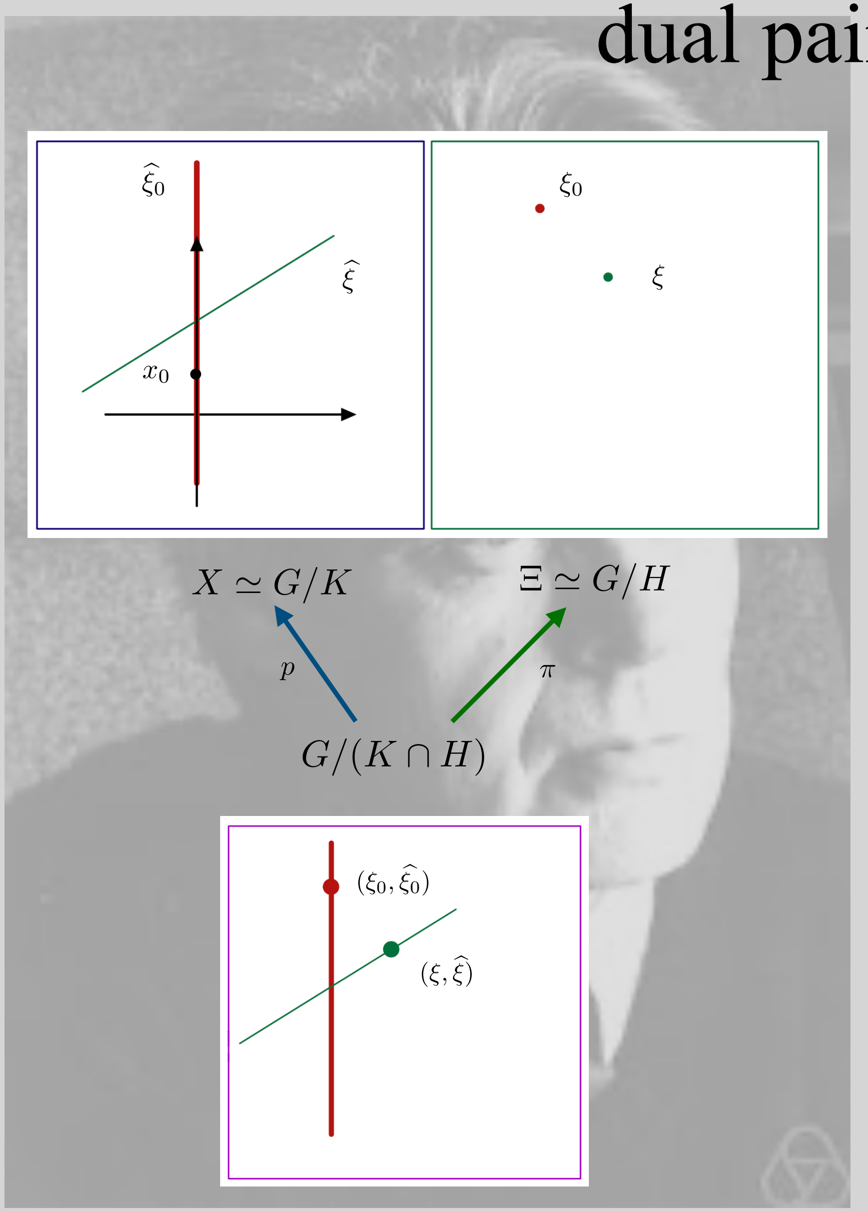
dual pairs - homogeneous spaces



$X \simeq G/K$ $\qquad$ $\Xi \simeq G/H$

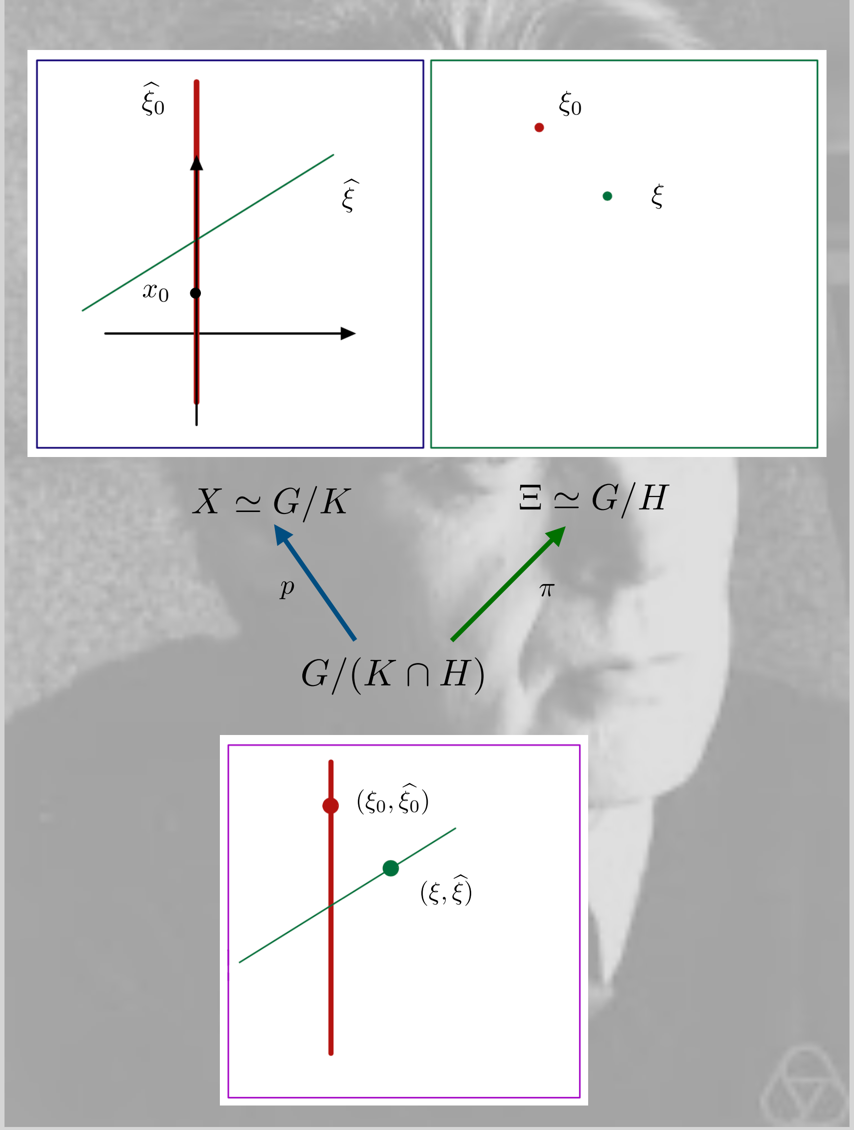
- G a lcsc topological group
- X, Ξ transitive G-spaces
- G acts transitively on X with action $x \mapsto g[x]$ with isotropy K at $x_0 \in X$ $\Big\}$ $X \simeq G/K$
- G acts transitively on Ξ with action $x \mapsto g.\xi$ with isotropy H at $\xi_0 \in \Xi$ $\Big\}$ $\Xi \simeq G/H$

dual pairs - homogeneous spaces



- G a lcsc topological group
- K and H two closed subgroups

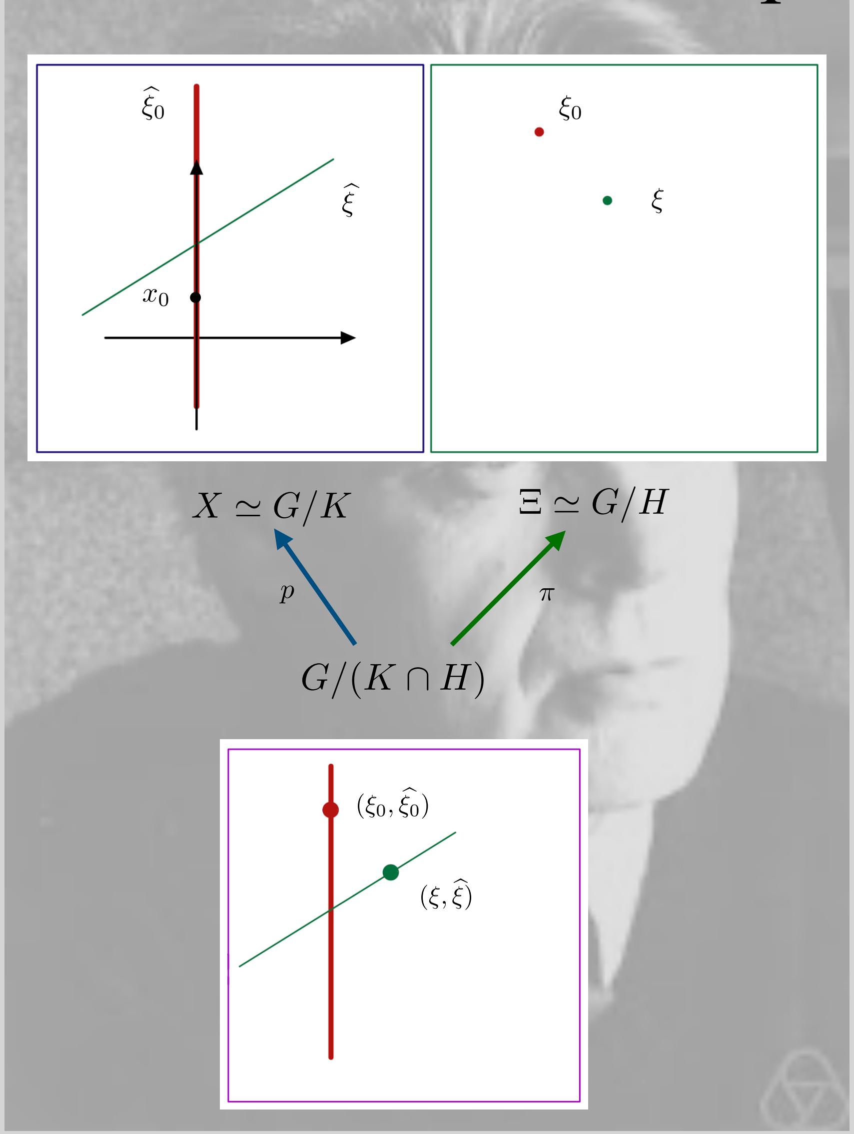
dual pairs - homogeneous spaces



- G a lcsc topological group
- K and H two closed subgroups
- the points $x = g_1 K,$ and $\xi = g_2 H$ are *incident* if

$$g_1 K \cap g_2 H \neq \emptyset, \qquad \text{(Chern, 1942)}$$

dual pairs - homogeneous spaces



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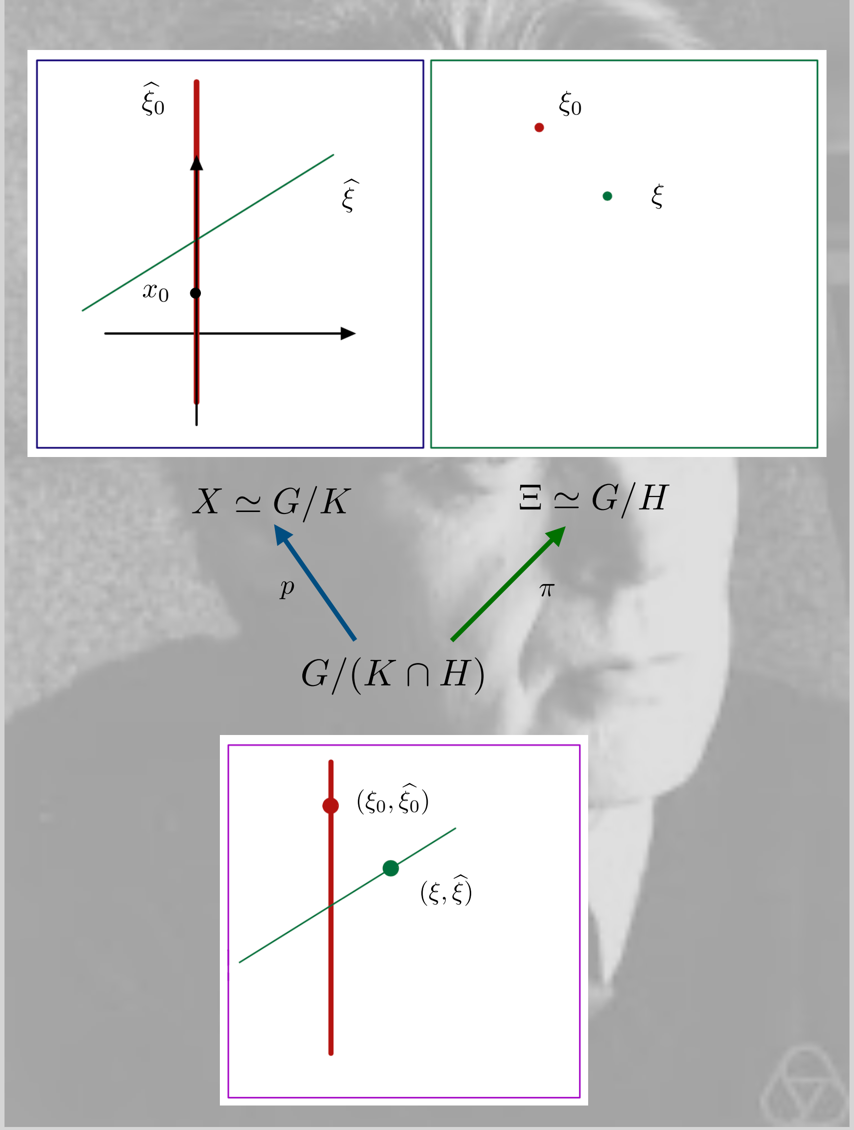
- $\widehat{\xi} = \{x \in X : x \text{ and } \xi \text{ are incident}\}$
- $\check{x} = \{\xi \in \Xi : x \text{ and } \xi \text{ are incident}\}$

dual pairs - homogeneous spaces



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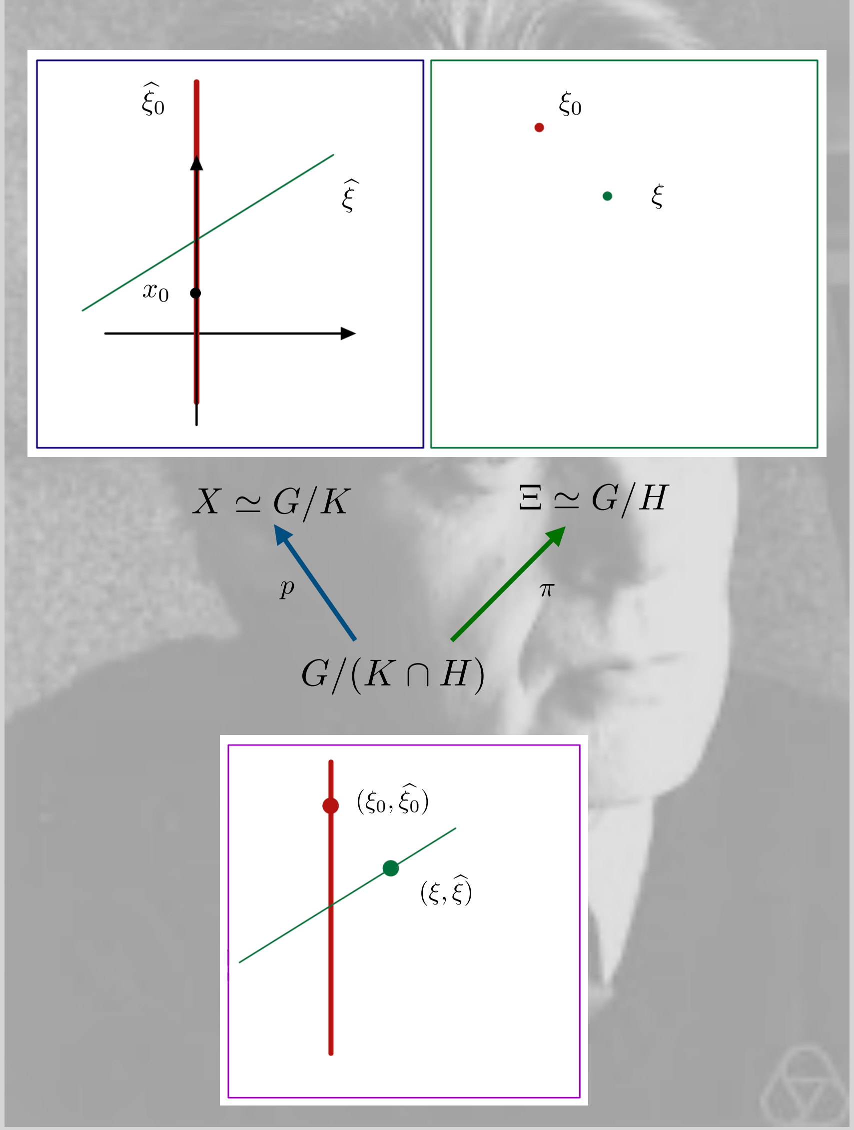
- (X, Ξ) is a dual pair if the maps $x \mapsto \check{x}$, $\xi \mapsto \widehat{\xi}$ are both injective

dual pairs - homogeneous spaces



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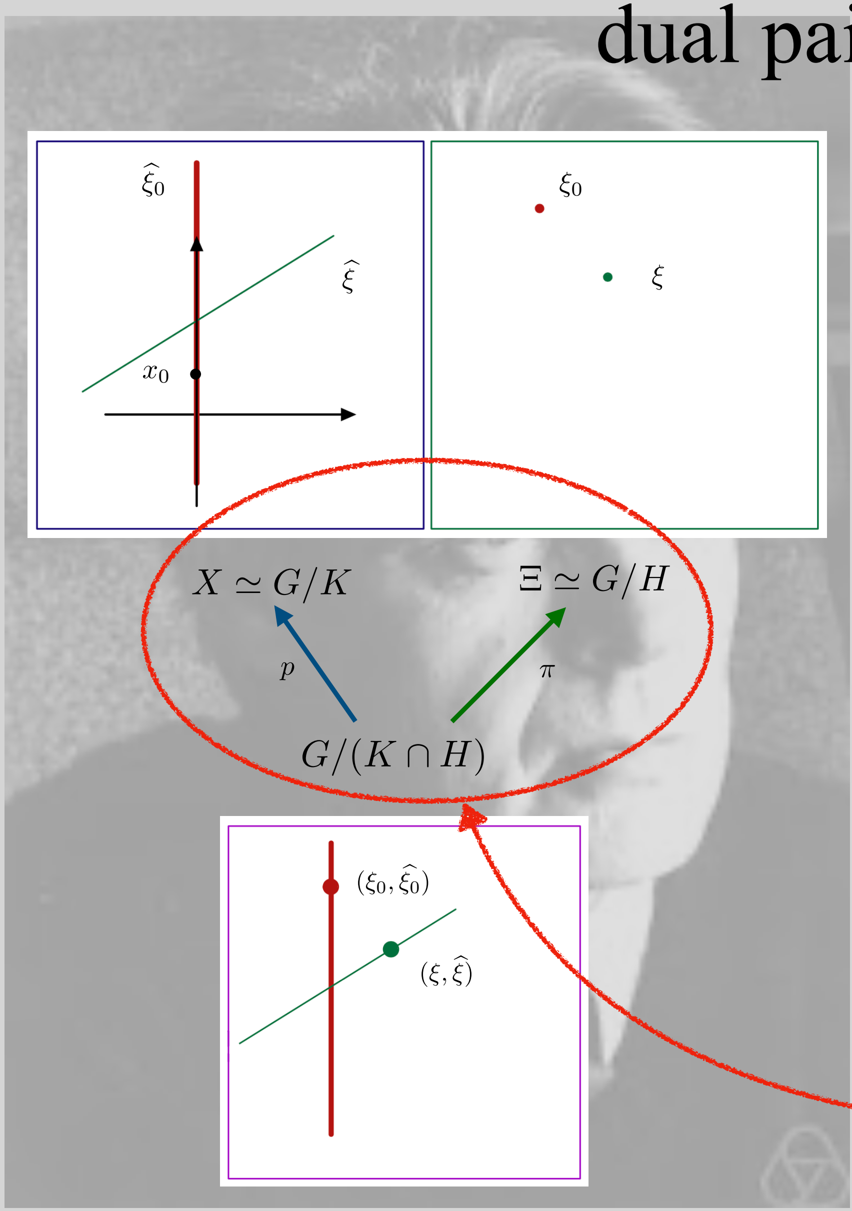
- $G/(K \cap H) \simeq \{(x, \xi) : x \text{ and } \xi \text{ are incident}\}$

dual pairs - homogeneous spaces



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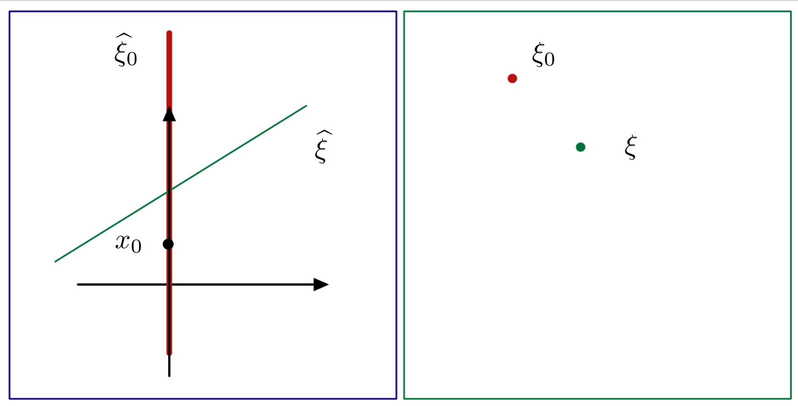
- double fibration

dual pairs with measures



$X \simeq G/K$ $\qquad$ $\Xi \simeq G/H$

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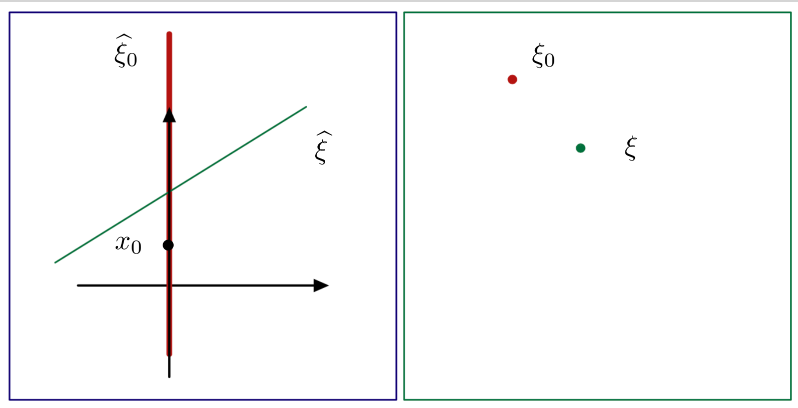
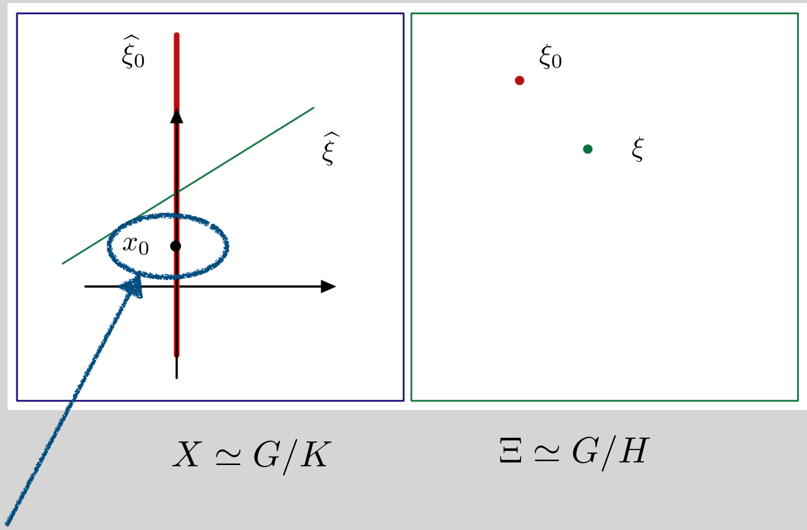
dual pairs with measures



$X \simeq G/K$ $\quad\quad$ $\Xi \simeq G/H$

- G a lcsc topological group
- X, Ξ transitive G-spaces
- Assumption: X, Ξ admit relatively G-invariant measures $\mathrm{d}x$, $\mathrm{d}\xi$ with positive characters $\alpha\colon G \to (0, +\infty)$ and $\beta\colon G \to (0, +\infty)$
- $\displaystyle\int_X f(g^{-1}[x])\mathrm{d}x = \alpha(g)\int_X f(x)\mathrm{d}x, \qquad f \in L^1(X, \mathrm{d}x),\ g \in G$
- $\displaystyle\int_\Xi f(g^{-1}.\xi)\mathrm{d}\xi = \beta(g)\int_\Xi f(\xi)\mathrm{d}\xi, \qquad f \in L^1(\Xi, \mathrm{d}\xi),\ g \in G$

dual pairs with measures



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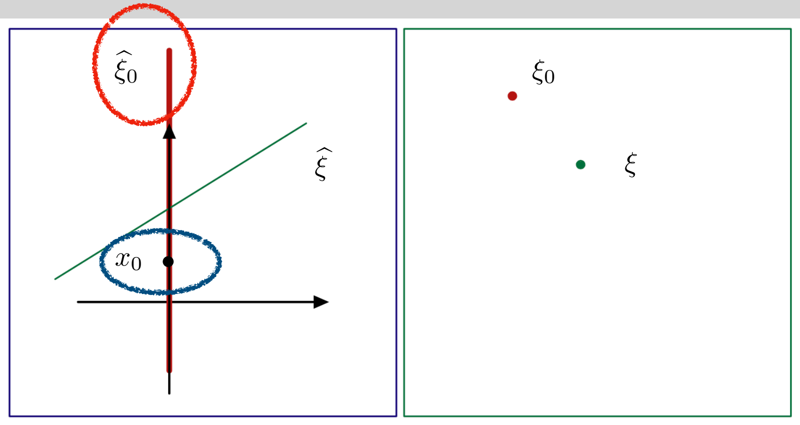
dual pairs with measures



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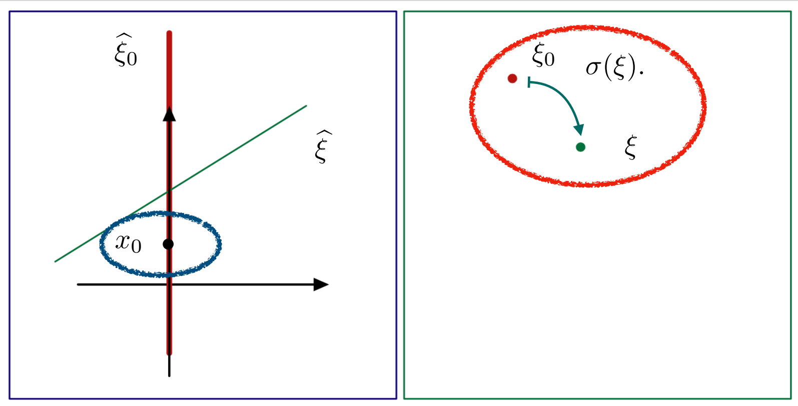
- $\widehat{\xi_0} = H[x_0] \subseteq X$

dual pairs with measures



$X \simeq G/K$ $\Xi \simeq G/H$

- $\widehat{\xi_0} = H[x_0] \subseteq X$
- $\sigma \colon \Xi \to G$ a Borel section

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- X, Ξ transitive G-spaces
- Assumption: X, Ξ admit relatively G-invariant measures $\mathrm{d}x$, $\mathrm{d}\xi$ with positive characters $\alpha \colon G \to (0, +\infty)$ and $\beta \colon G \to (0, +\infty)$
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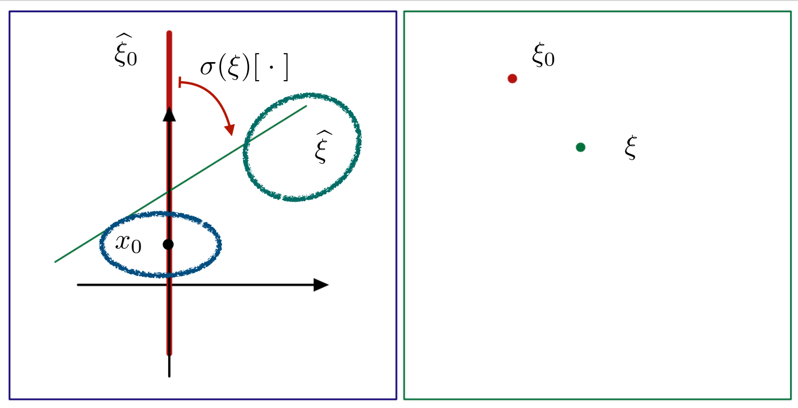
dual pairs - homogeneous spaces



$X \simeq G/K$ $\Xi \simeq G/H$

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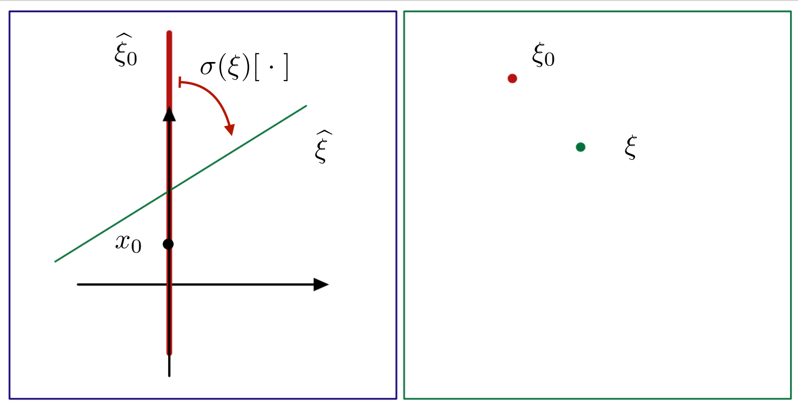

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$$\int_{\widehat{\xi}} f(x)\,\mathrm{d}m_\xi(x) = \int_{\widehat{\xi_0}} f(\sigma(\xi)[x])\,\mathrm{d}m_0(x)$$

Radon transform on dual pairs with measures

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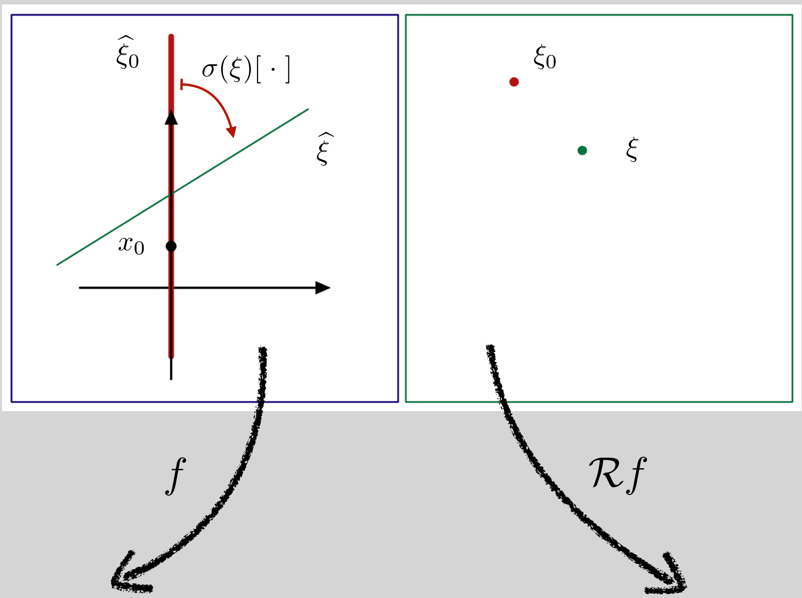

Radon transform on a G-dual pair with measures:

$$\mathcal{R}f(\xi) = \int_{\widehat{\xi}} f(x)\mathrm{d}m_\xi(x) = \int_{\widehat{\xi_0}} f(\sigma(\xi)[x])\mathrm{d}m_0(x)$$

quasi-regular representations

Assumptions on the quasi-regular representations:

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- there exists a non-trivial π-invariant $\mathcal{A} \subseteq L^2(X, \mathrm{d}x)$ such that $\mathcal{R}f$ is well defined for all $f \in \mathcal{A}$ and the adjoint of the operator $\mathcal{R}\colon \mathcal{A} \to L^2(X, \mathrm{d}\xi)$ has non-trivial domain

the closure of the Radon transform

Theorem. (Alberti, Bartolucci, De Vito & FDM) $\mathcal{R}\colon \mathcal{A} \to L^2(\Xi, \mathrm{d}\xi)$ is closable and its closure $\overline{\mathcal{R}}$ is a densely defined operator satisfying:

$$\overline{\mathcal{R}}\pi(g) = \chi(g)^{-1}\hat{\pi}(g)\overline{\mathcal{R}},$$

$$\chi(g) = \alpha(g)^{1/2}\beta(g)^{-1/2}\gamma(g\sigma(g^{-1}.\xi_0))^{-1}.$$

Furthermore, $\overline{\mathcal{R}}$ is the unique closed extension of $\mathcal{R}$.

G. Alberti, F. Bartolucci, F. De Mari, E. De Vito, **Unitarization and Inversion Formulae for the Radon Transform between Dual Pairs.** *SIAM J. Math. Anal.*, 2019.

the unitarization theorem

Theorem. (Alberti, Bartolucci, De Vito & FDM) There exists a unique positive self-adjoint operator

$$\mathcal{I}\colon \mathrm{dom}(\mathcal{I}) \supset \mathrm{Im}(\overline{\mathcal{R}}) \to L^2(\Xi, \mathrm{d}\xi)$$

with the property

$$\hat{\pi}(g)\mathcal{I}\hat{\pi}(g)^{-1} = \chi(g)^{-1}\mathcal{I}, \qquad g \in G$$

such that the composite operator $\mathcal{I}\overline{\mathcal{R}}$ extends to a unitary operator $\mathcal{Q}\colon L^2(X, \mathrm{d}x) \to L^2(\Xi, \mathrm{d}\xi)$ intertwining π and $\hat{\pi}$, namely:

$$\hat{\pi}(g)\mathcal{Q}\pi(g)^{-1} = \mathcal{Q}, \qquad g \in G.$$

G. Alberti, F. Bartolucci, F. De Mari, E. De Vito, **Unitarization and Inversion Formulae for the Radon Transform between Dual Pairs.** *SIAM J. Math. Anal.*, 2019.

the inversion theorem

Assumption: π is square integrable, that is: there exists $\psi \in L^2(X, \mathrm{d}x)$, called and *admissible vector*, such that the *voice transform:*

$$f \mapsto V_\psi f, \quad V_\psi f(g) = \langle f, \pi(g)\psi \rangle$$

is an isometry of $L^2(X, \mathrm{d}x)$ into $L^2(G, \mathrm{d}g)$. In this case:

$$f = \int_G \langle f, \pi(g)\psi \rangle \pi(g)\psi \, \mathrm{d}g \qquad \text{(weakly)}$$

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$$f = \int_G \chi(g) \langle \overline{\mathcal{R}} f, \widehat{\pi}(g)\Psi \rangle \pi(g)\psi \, \mathrm{d}g$$

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G. Alberti, F. Bartolucci, E. De Vito & FDM **Unitarization and Inversion Formulae for the Radon Transform between Dual Pairs.** *SIAM J. Math. Anal.*, 2019.

the inversion theorem - moral proof

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G. Alberti, F. Bartolucci, E. De Vito & FDM **Unitarization and Inversion Formulae for the Radon Transform between Dual Pairs.** *SIAM J. Math. Anal.*, 2019.

example: the *shearlet* group

the group generated by translations, parabolic dilations and shearing in $\mathbb{R}^2$

$$A_a = a \begin{bmatrix} 1 & 0 \\ 0 & |a|^{-1/2} \end{bmatrix}, \; a \in \mathbb{R}^* \qquad S_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, \; s \in \mathbb{R}$$

K. Guo, D. Labate, **Optimally sparse multidimensional representation using shearlets,** *SIAM journal on mathematical analysis,* 2007

G. Kutyniok, D. Labate, **Shearlets.** *Applied and Numerical Harmonic Analysis,* Springer, 2012

F. Colonna, G. Easley, K. Guo, D. Labate, **Radon transform inversion using the shearlet representation.** *ACHA*, 2010

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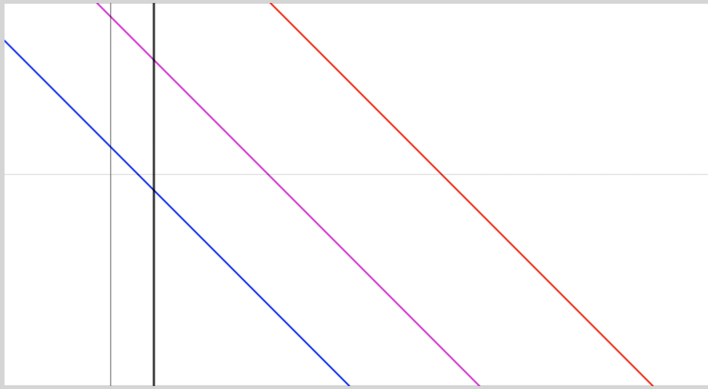
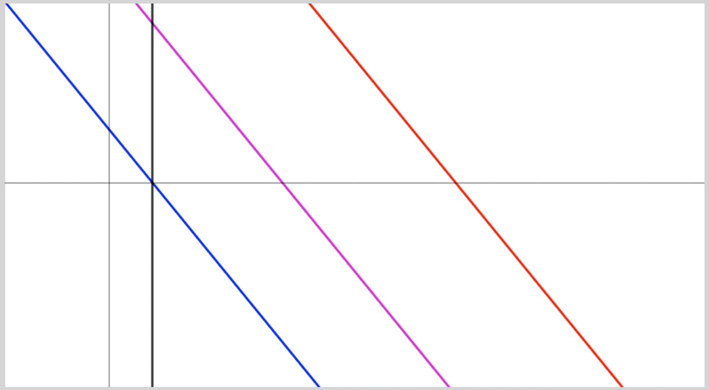
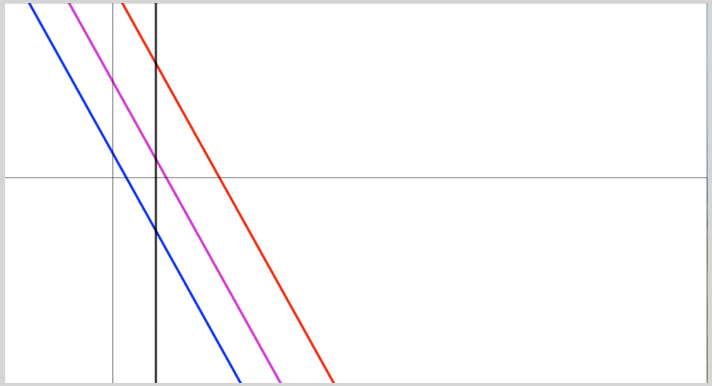


$(b,0,1).(v,t) = \big(v,(1,v)\cdot b\big)$



$(0,s,1).(v,t) = \big(v-s,t\big)$



$(0,0,a).(v,t) = \big(|a|^{-1/2}v,\frac{t}{a}\big)$

example: the *shearlet* group

- $f = \int_{\mathbb{R}^2} \int_{\mathbb{R}} \int_{\mathbb{R}^*} \mathcal{S}_\psi f(b, s, a) \psi_{b,s,a} \, \frac{\mathrm{d}b \, \mathrm{d}s \, \mathrm{d}a}{|a|^3}$

F. Colonna, G. Easley, K. Guo, D. Labate, R**adon transform inversion using the shearlet representation.**
ACHA, 2010.

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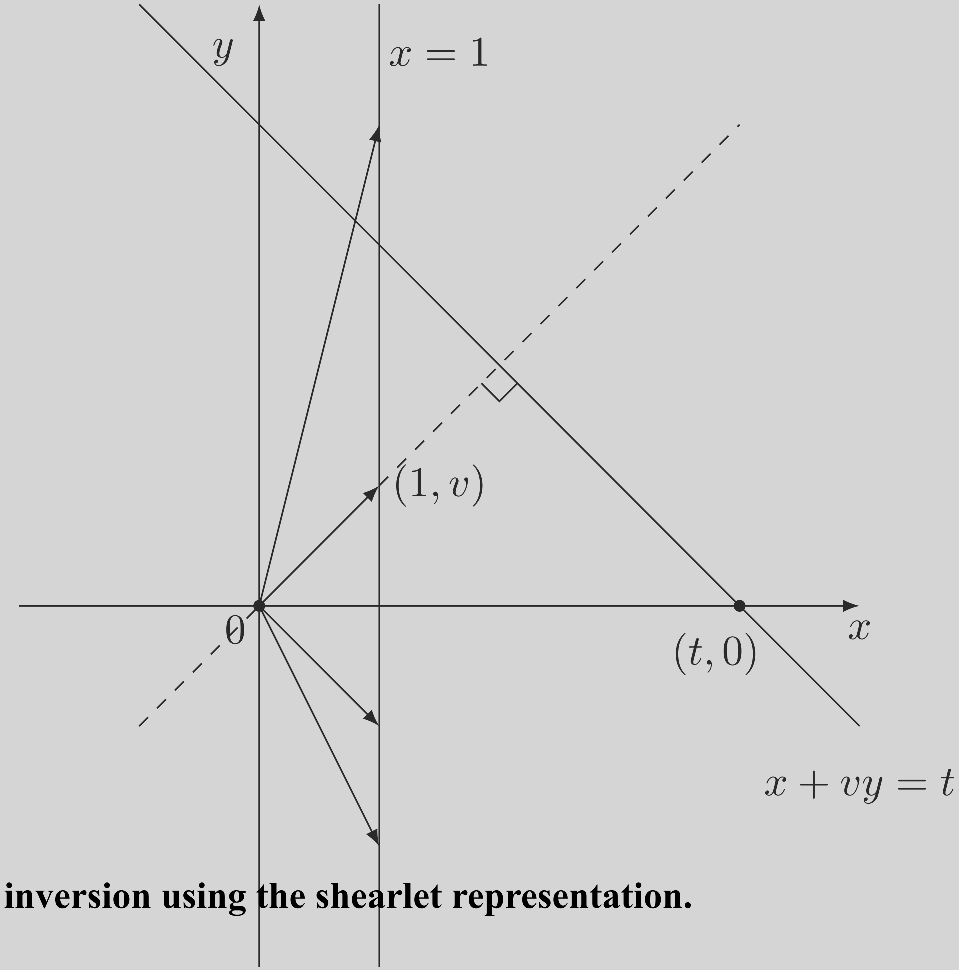

F. Colonna, G. Easley, K. Guo, D. Labate, **Radon transform inversion using the shearlet representation.** *ACHA*, 2010.

F. Bartolucci, E. De Vito, F. Odone & FDM **The Radon transform intertwines wavelets and shearlets.** *ACHA*, 2019.

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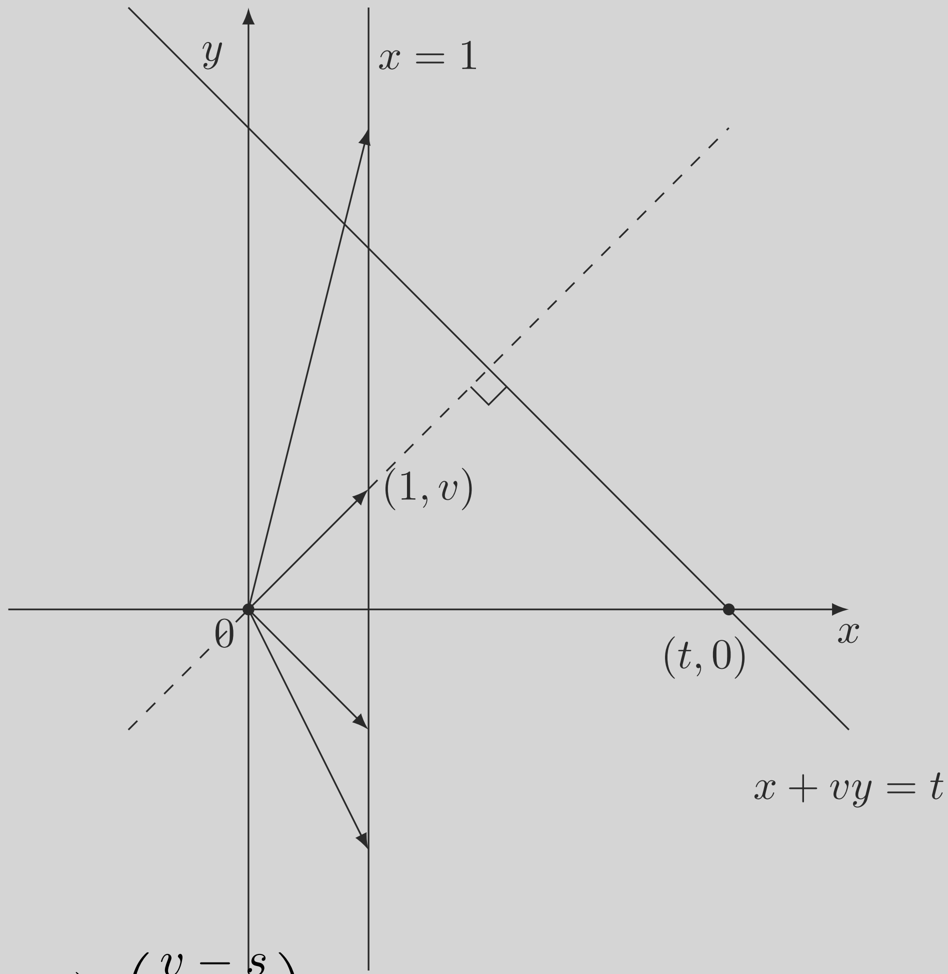

- $\mathcal{S}_\psi f((x,y),s,a) = |a|^{-3/4}\int_{\mathbb{R}} W_\chi\Big(\mathcal{R}f(v,\bullet)\Big)(x+vy,a)\phi\Big(\frac{v-s}{|a|^{1/2}}\Big)\mathrm{d}v$

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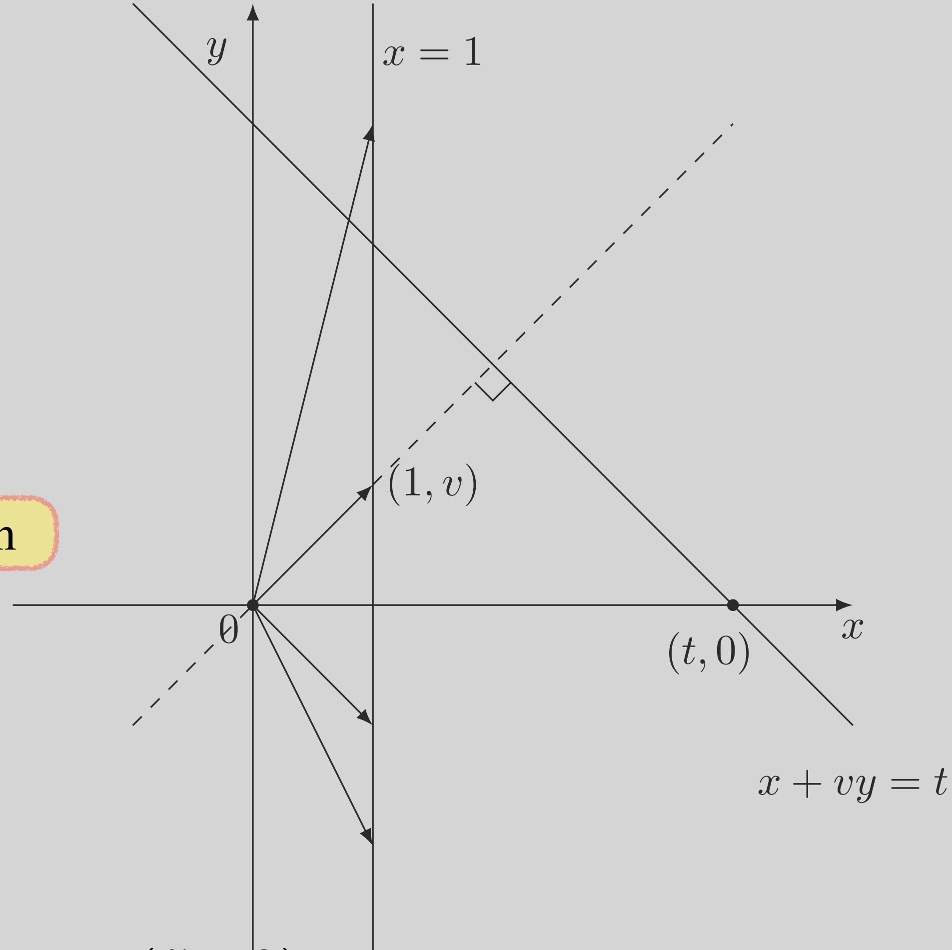

1. Radon transform

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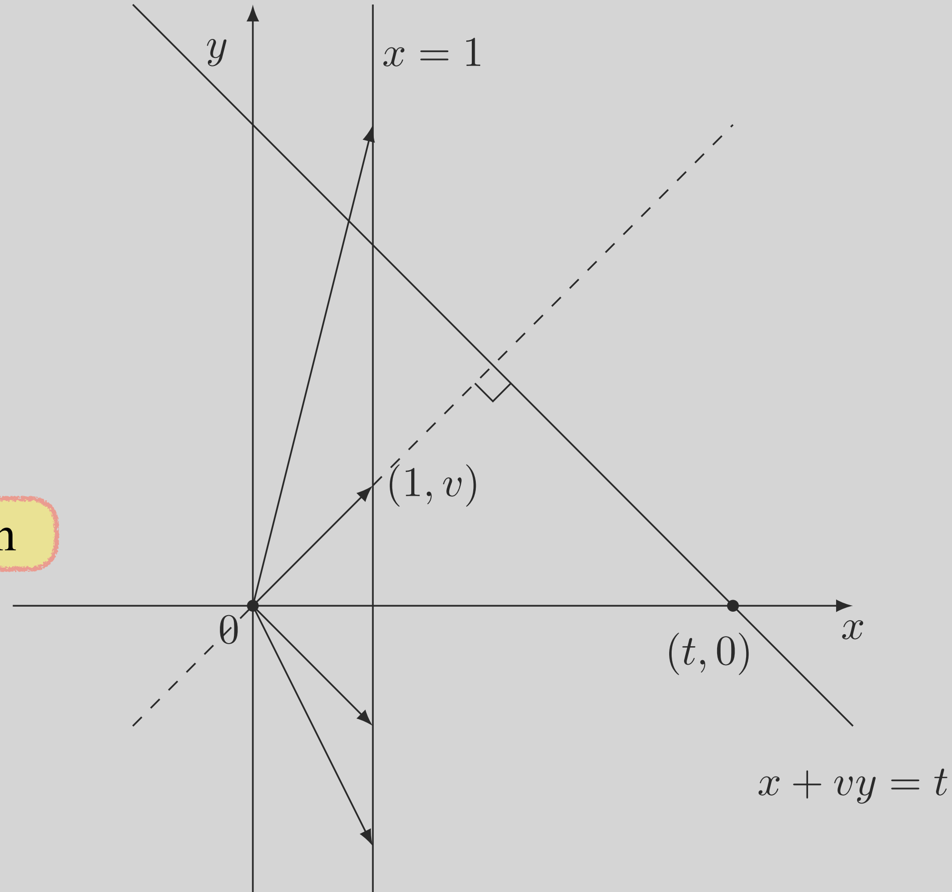

1. Radon transform

2. a 1D wavelet transform

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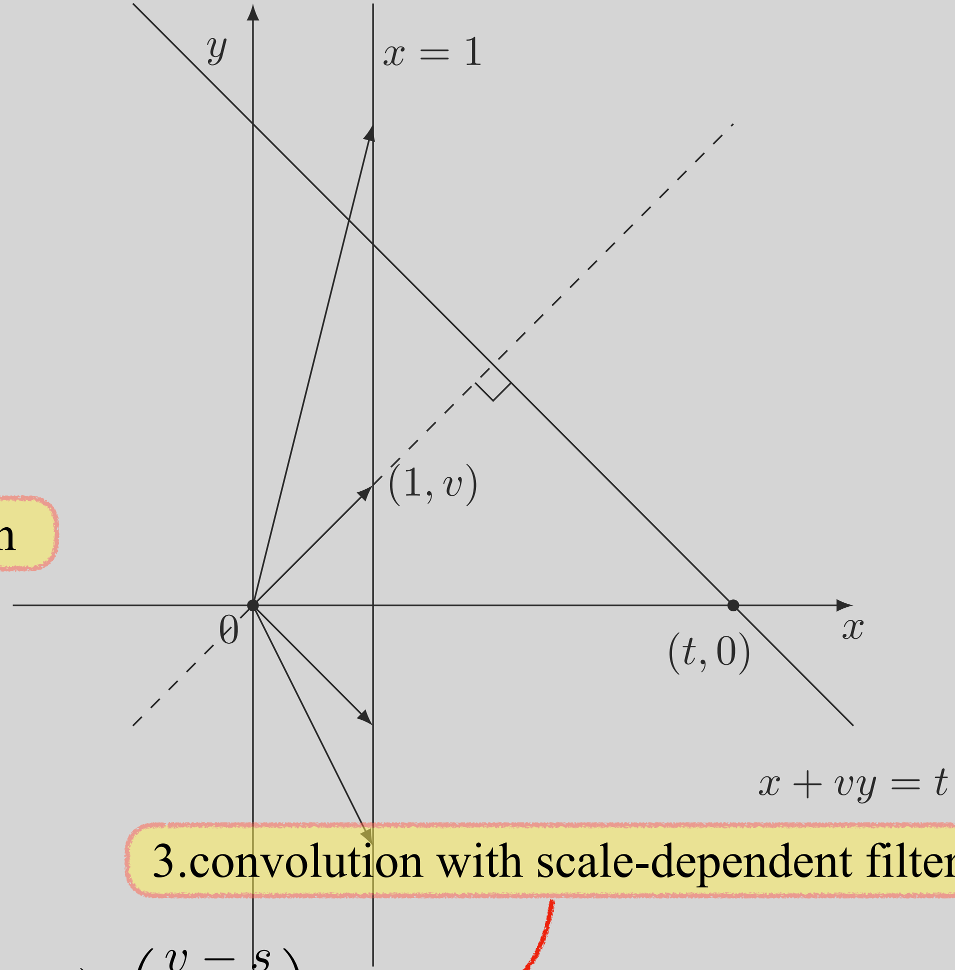

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1. Radon transform

2. a 1D wavelet transform

3.convolution with scale-dependent filter

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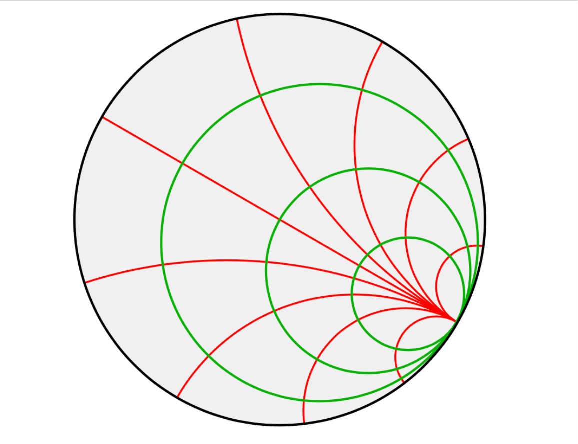
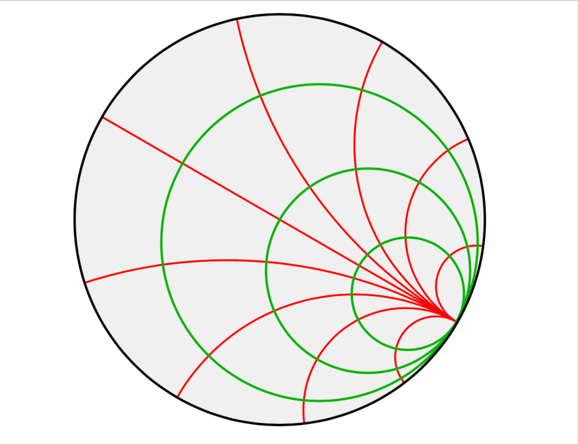
back to the origins: symmetric spaces?

… after all Helgason's theorem, proved in Euclidean setting should hold true for symmetric spaces…



Theorem (S. Helgason) The map $f \mapsto \mathbf{I}^{-\dot{\alpha}}\mathcal{R}f$ extends to a unitary operator $\mathcal{Q}\colon L^2(\mathbb{R}^d) \to L^2_{\mathrm{even}}(S^{d-1} \times \mathbb{R})$.

back to the origins: symmetric spaces?



- $X = G/K$ a noncompact symmetric space
- $\Xi = G/MN$ the space of horocycles in X
- $\mathcal{R}$ the horocyclic Radon transform

- the unitary quasi-regular representations $(\pi, L^2(X, \mathrm{d}x))$ and $(\widehat{\pi}, L^2(\Xi, \mathrm{d}\xi))$, namely:

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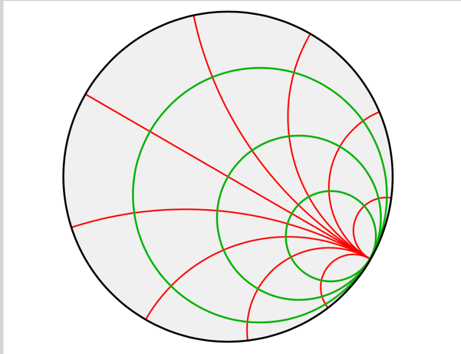
back to the origins: symmetric spaces?



- $X = G/K$ a noncompact symmetric space
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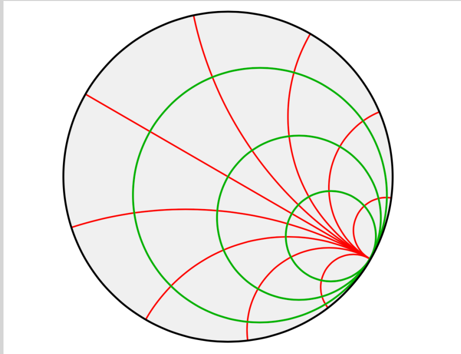

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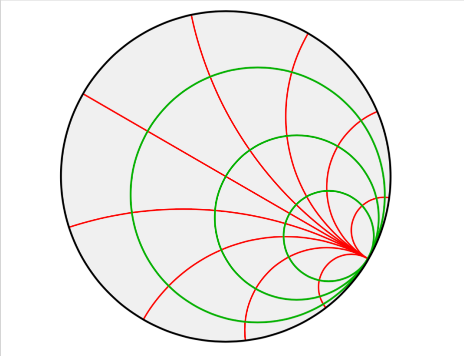

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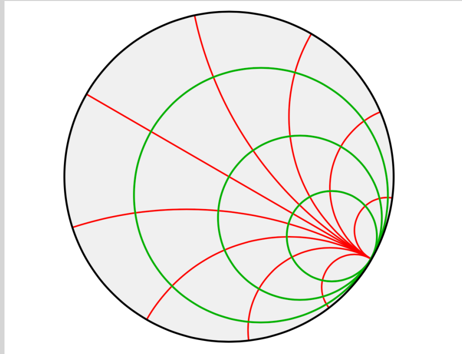

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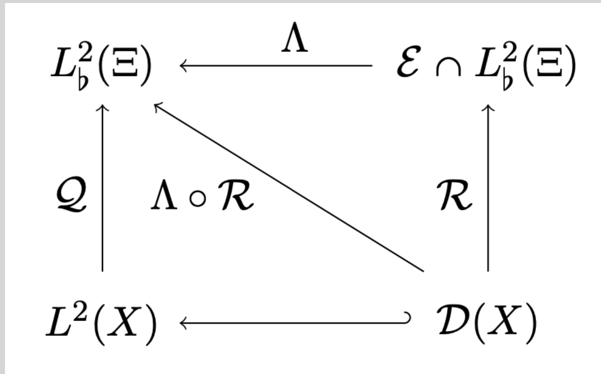
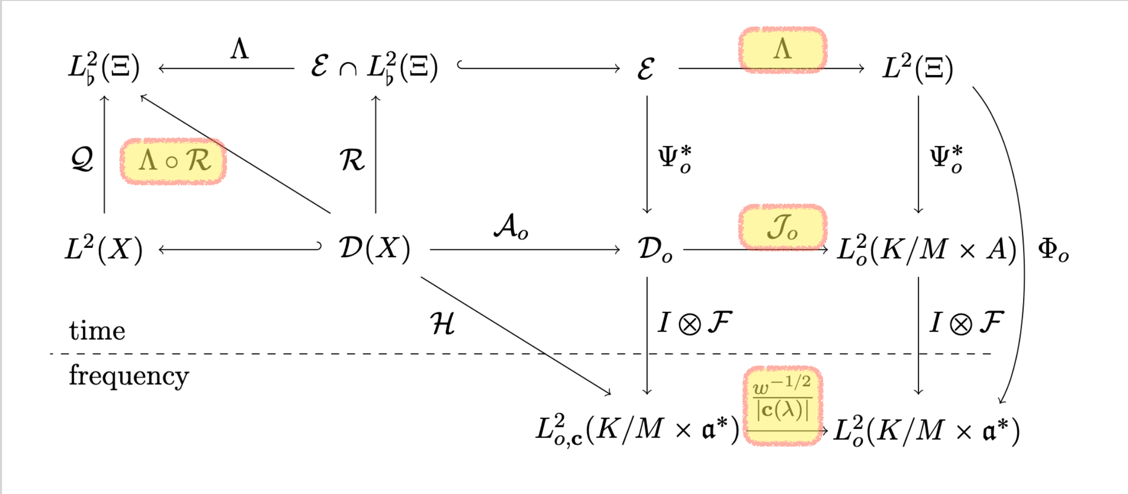


the unitarization theorem



Theorem 7. The composite operator $\Lambda\mathcal{R}$ extends to a unitary operator:

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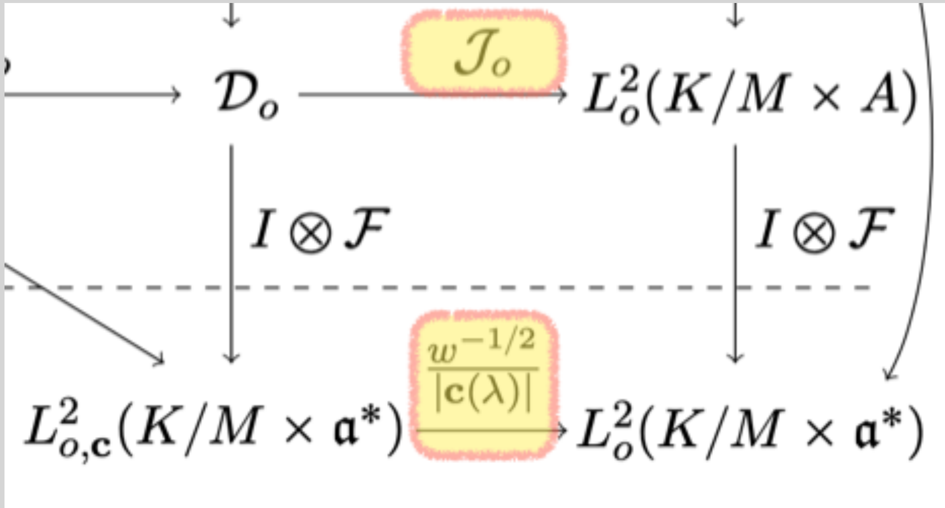

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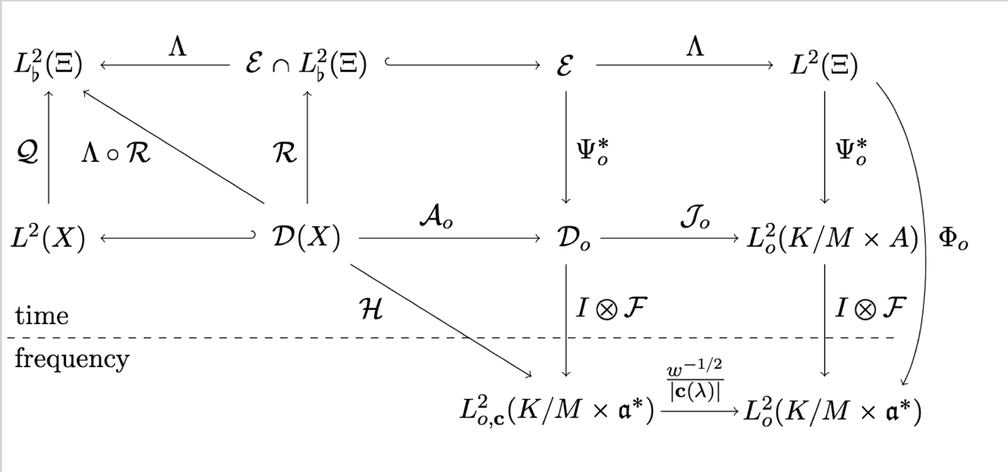
unitarization: symmetric spaces and homogeneous trees



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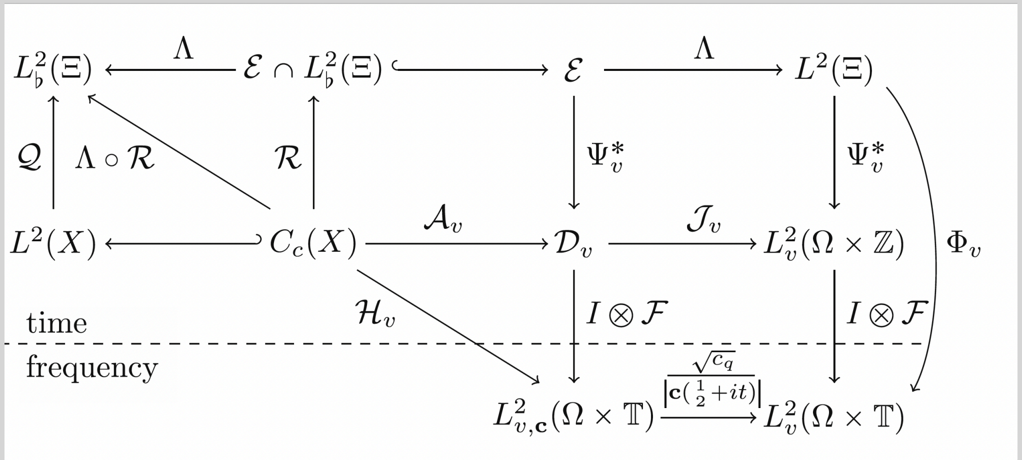

homogeneous trees

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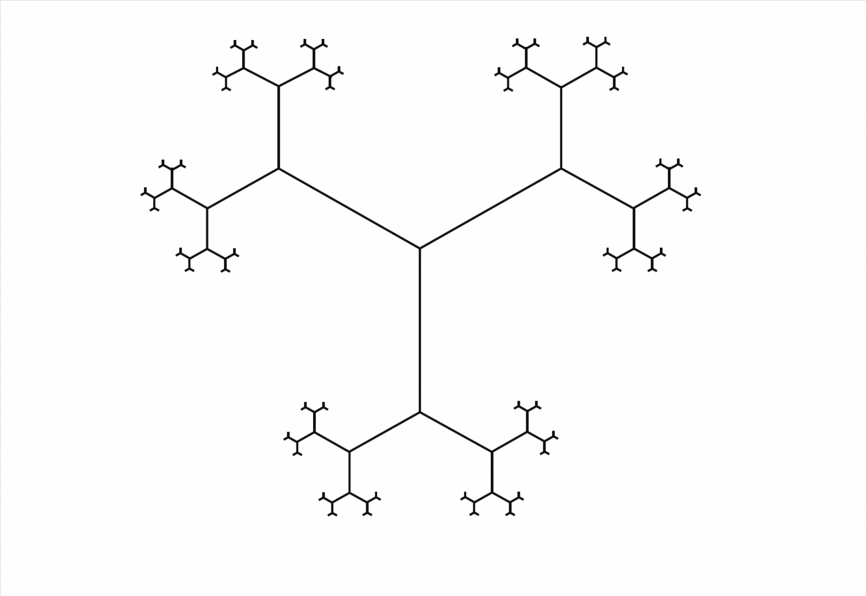

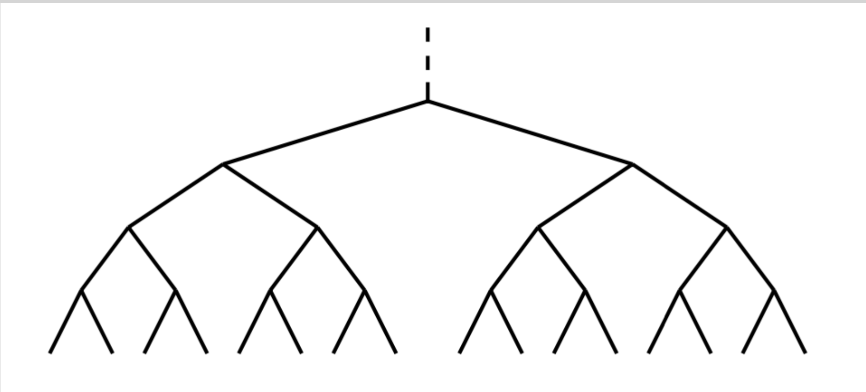

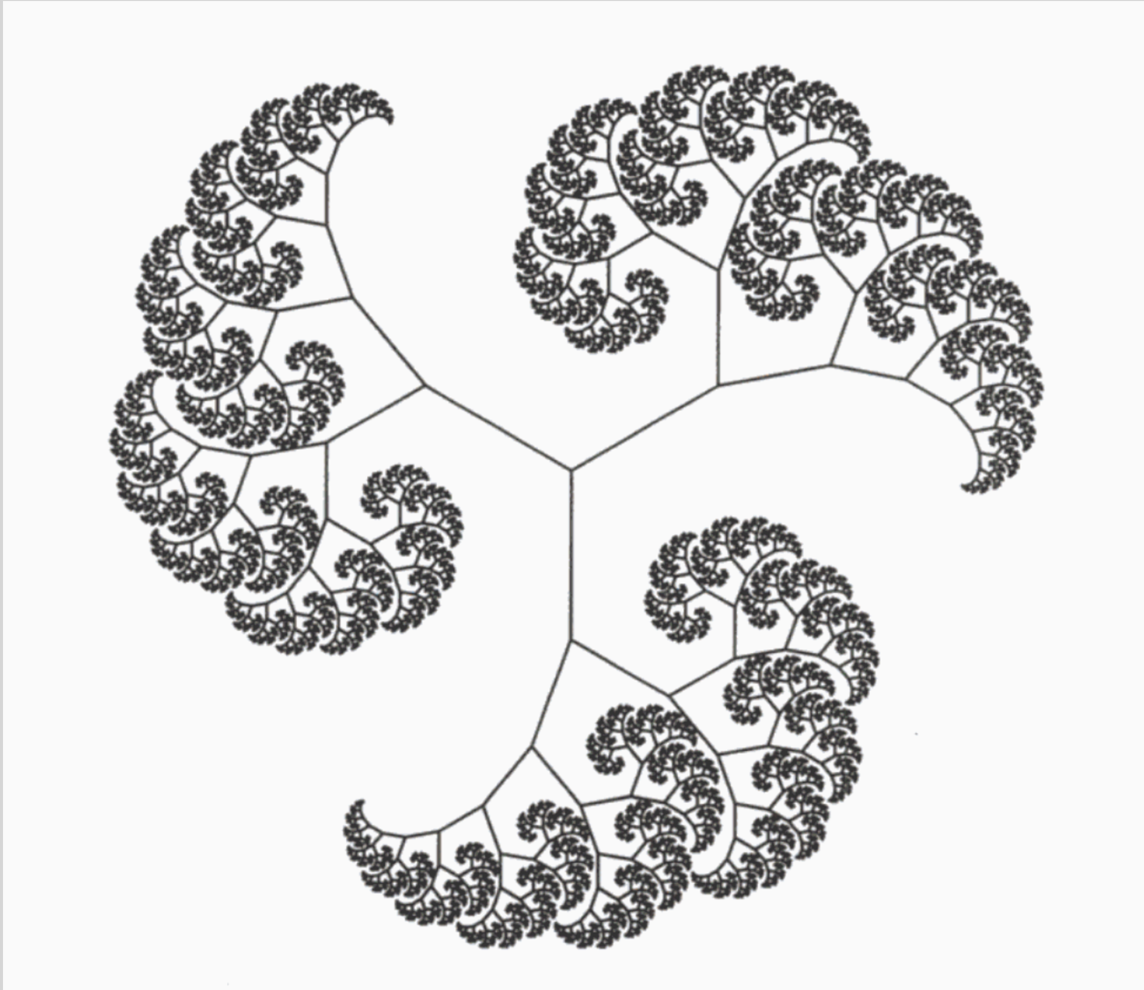

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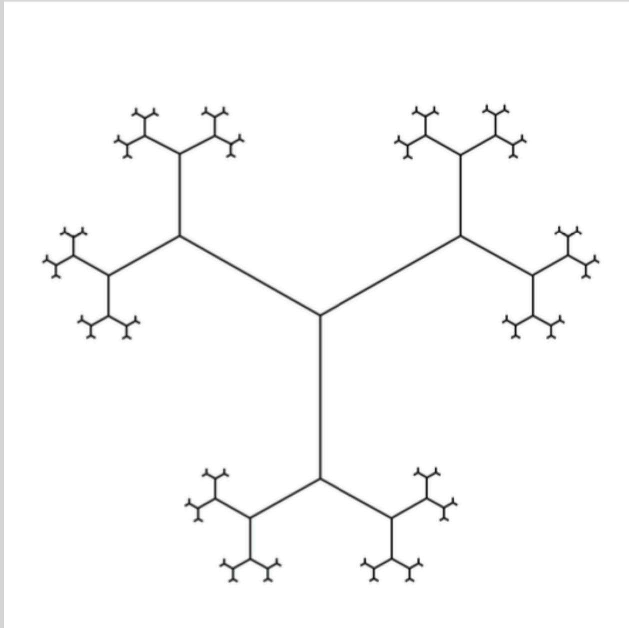

these are not trees!

… Tuesday morning, here

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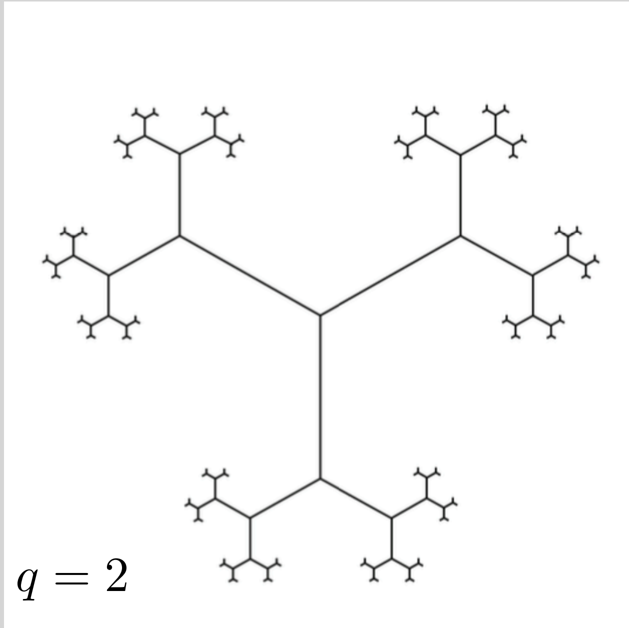

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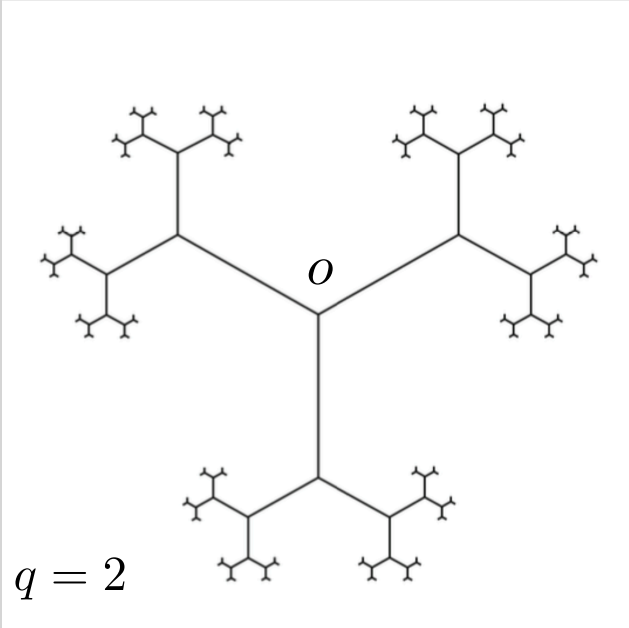

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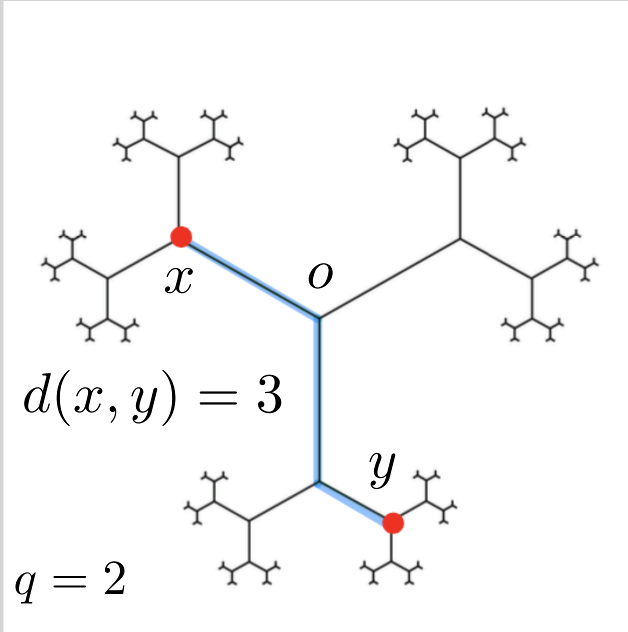

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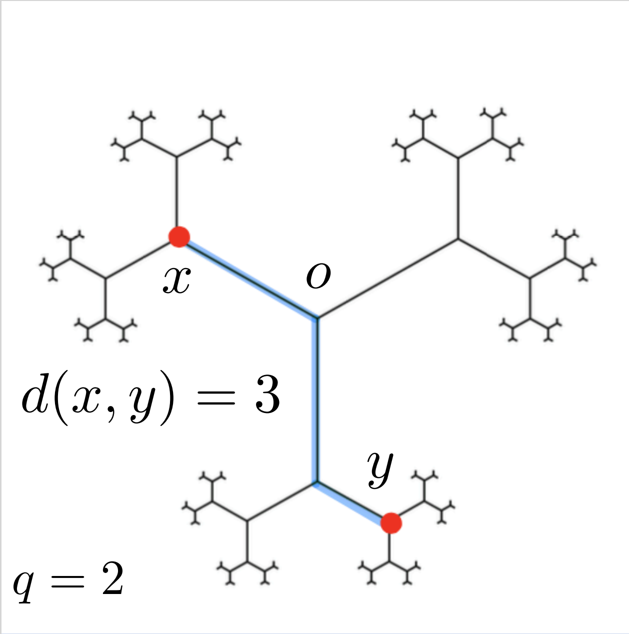

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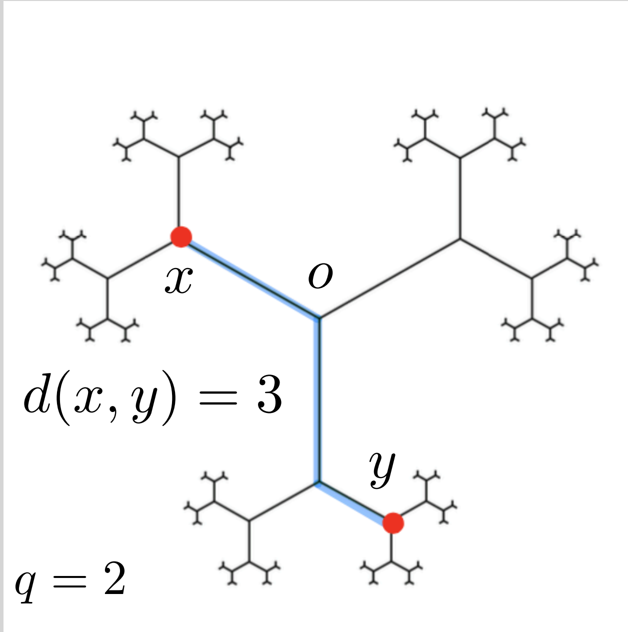

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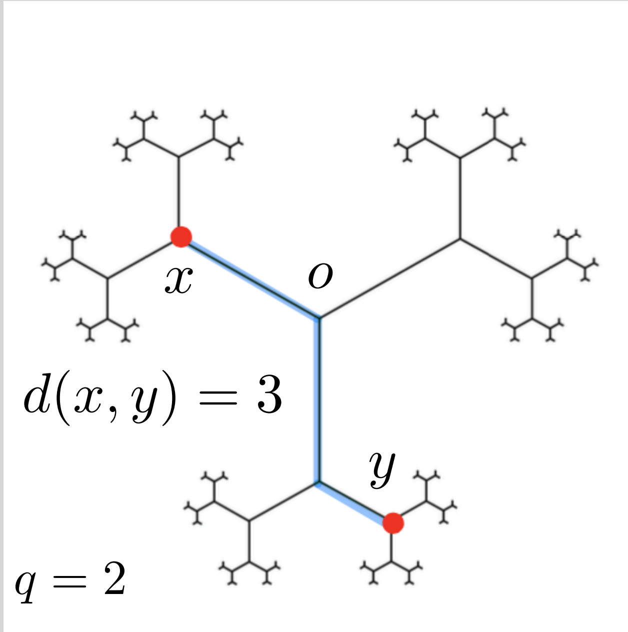
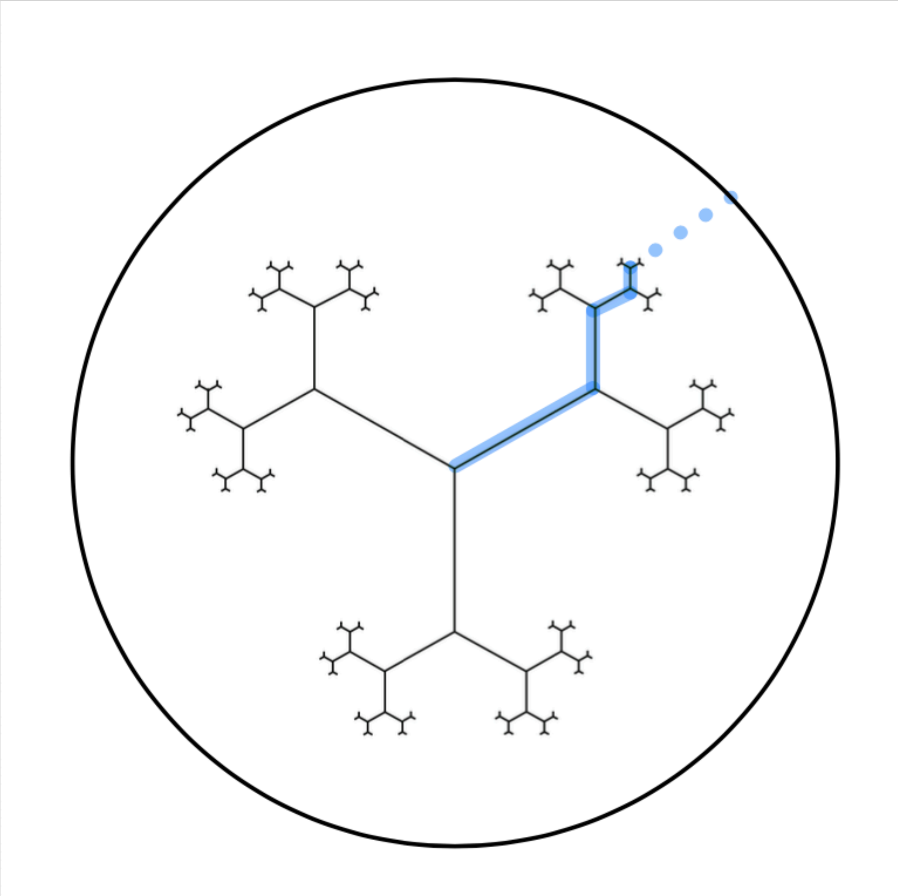


the boundary of a tree



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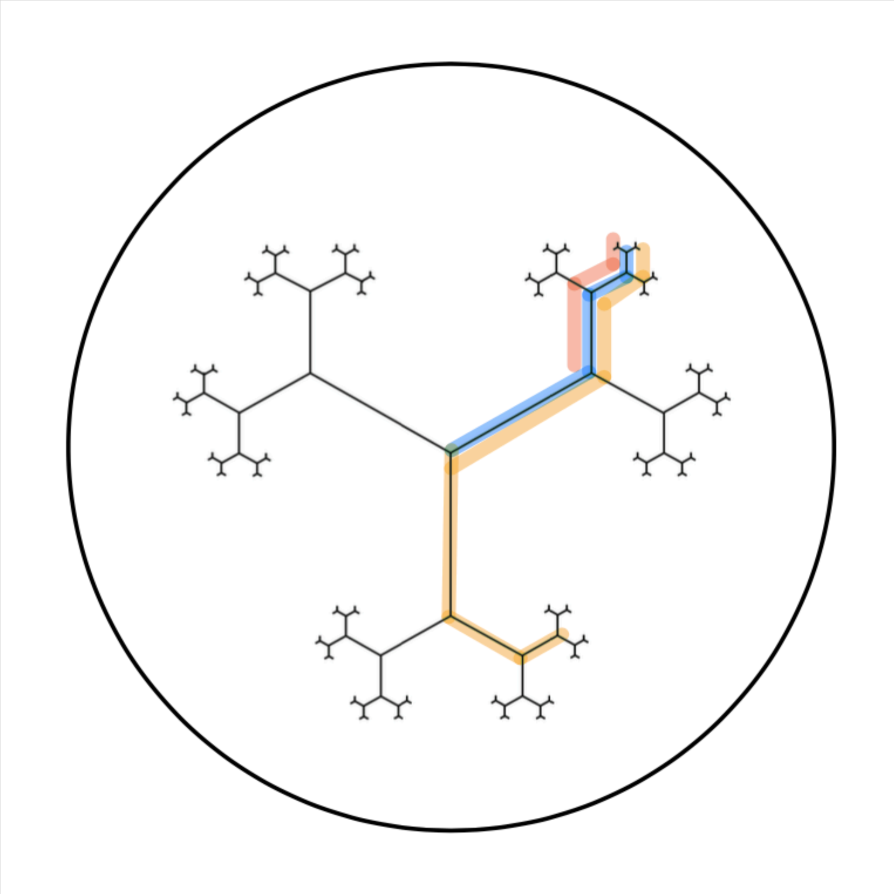

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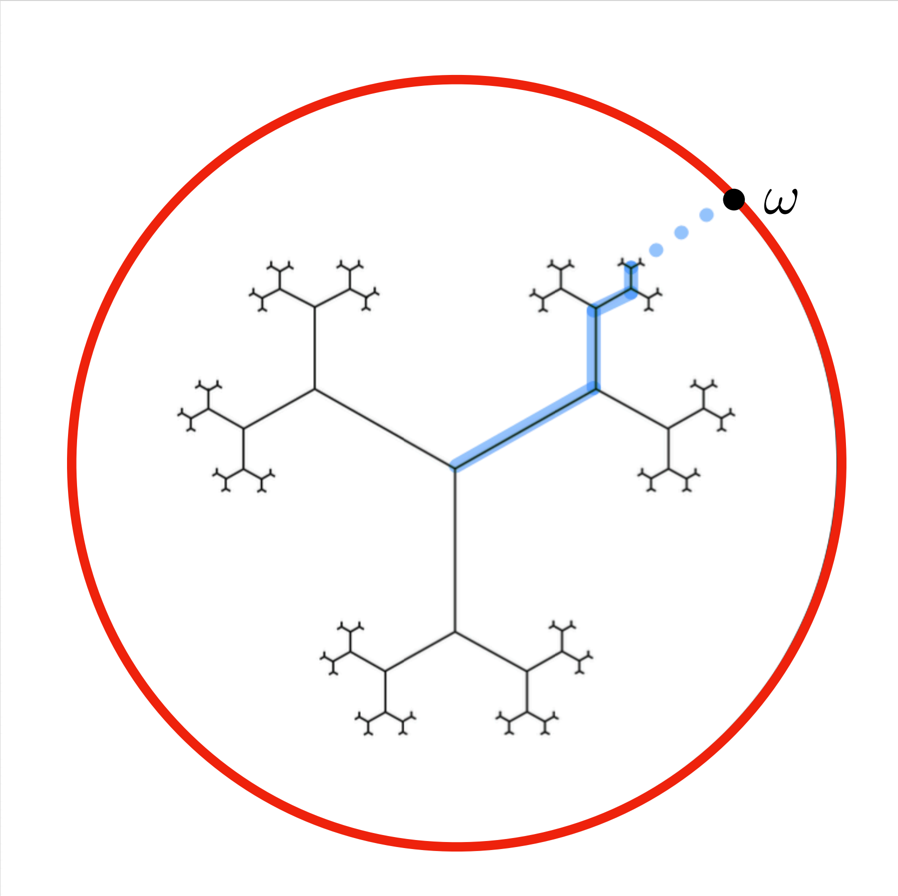

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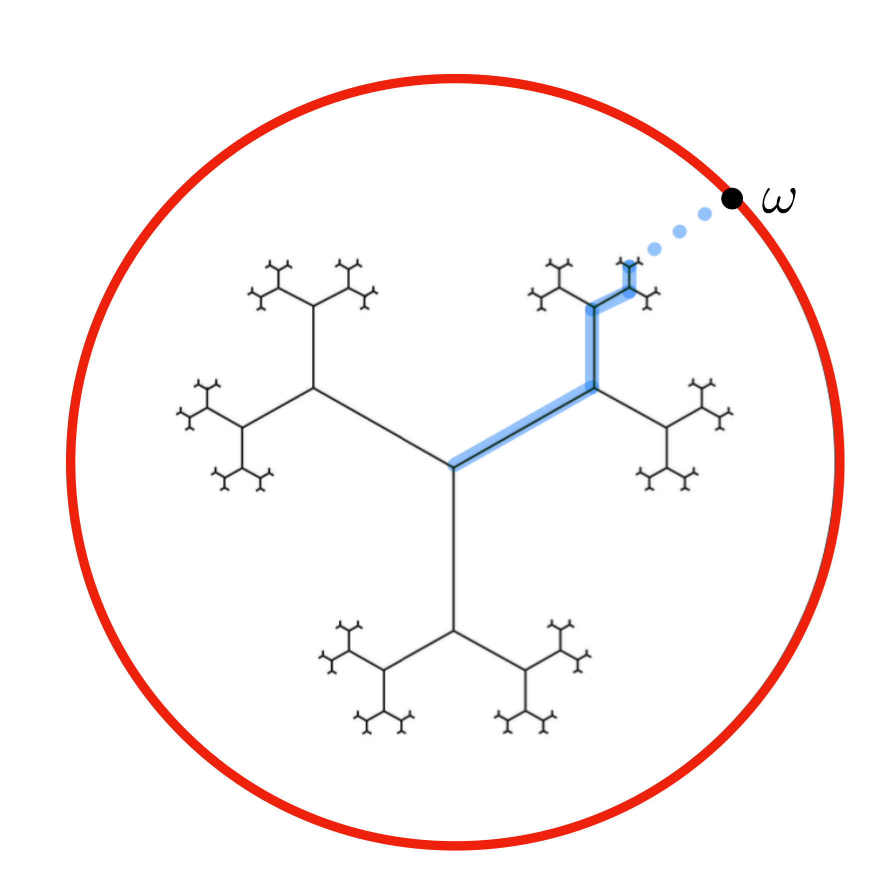

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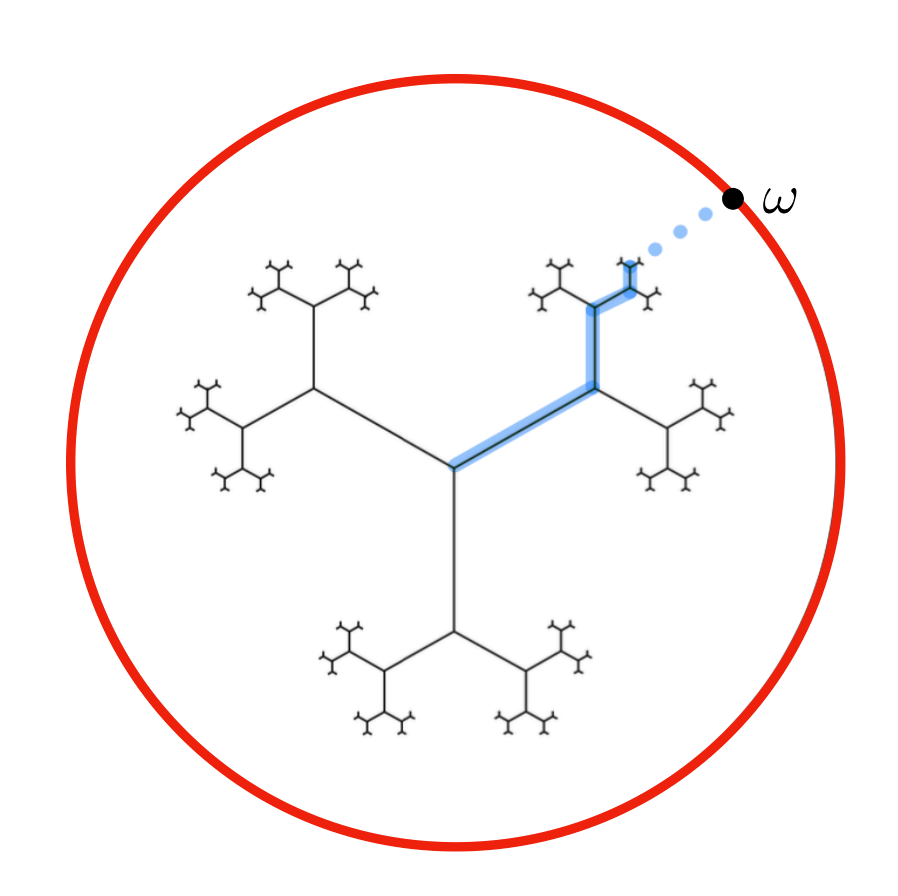

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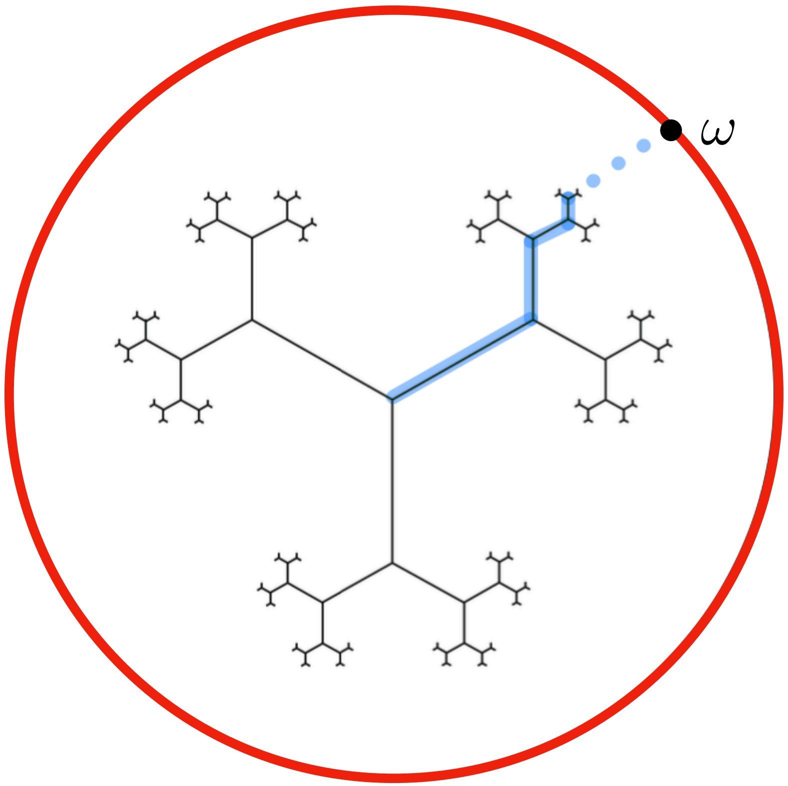

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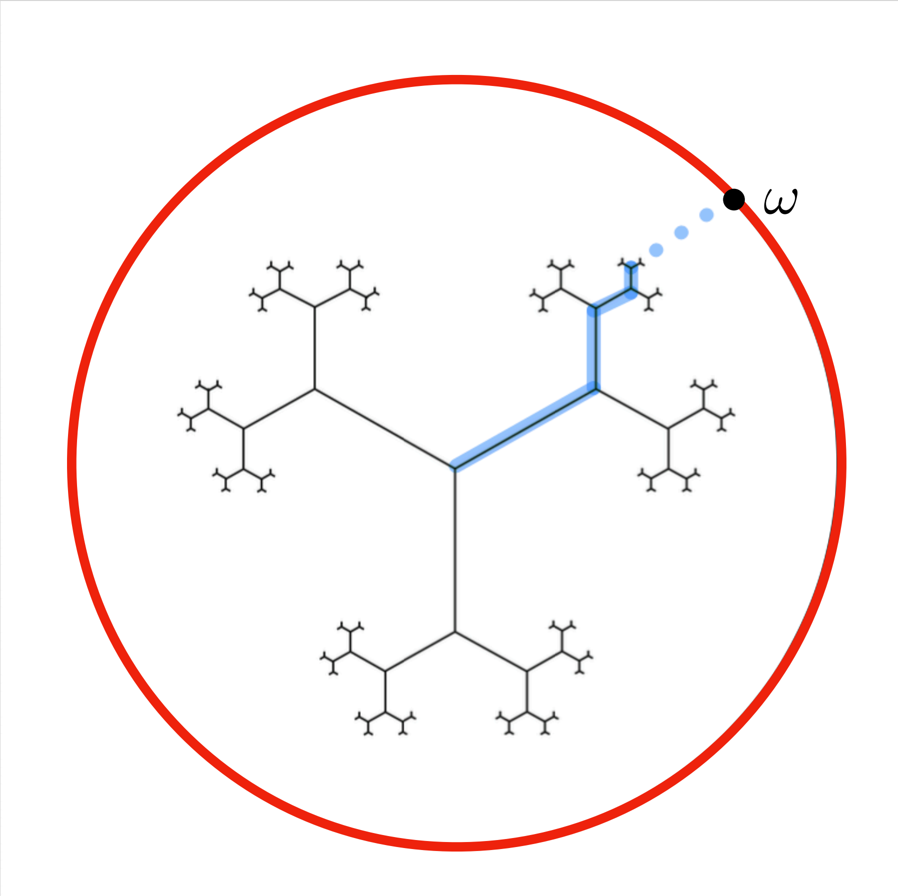

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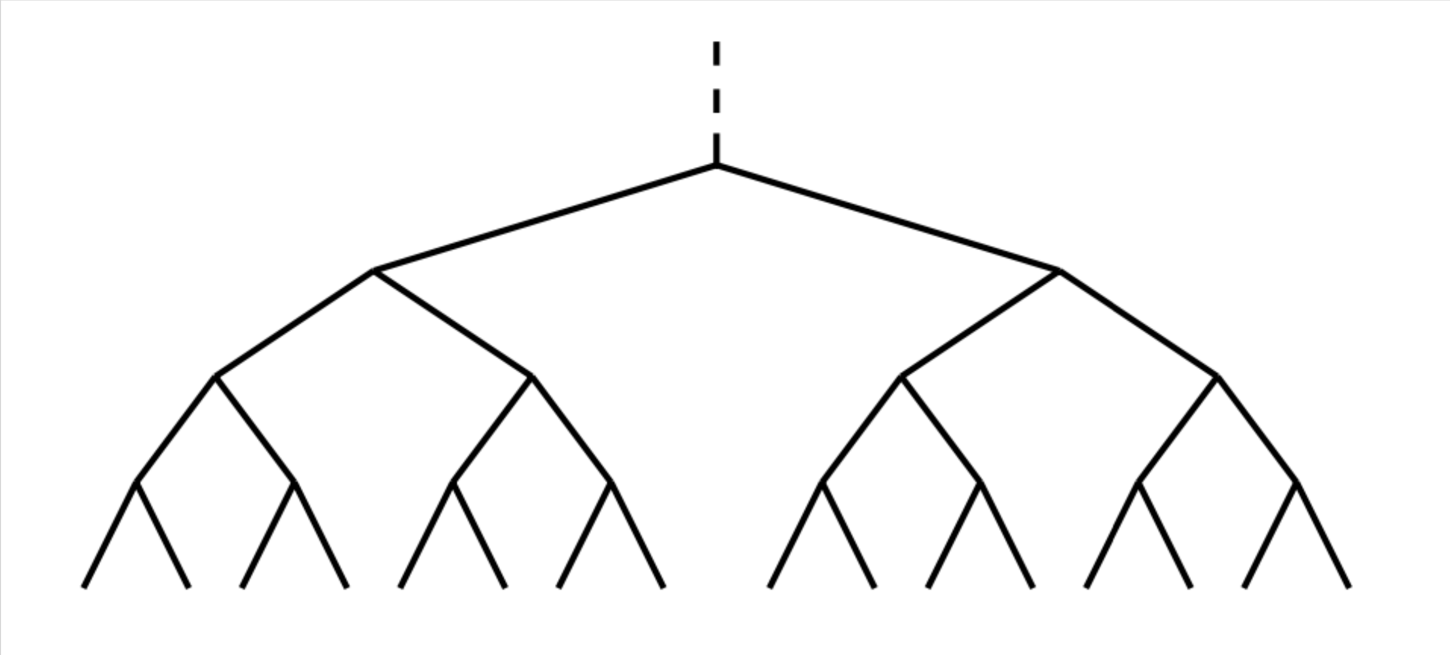
horocycles in a tree



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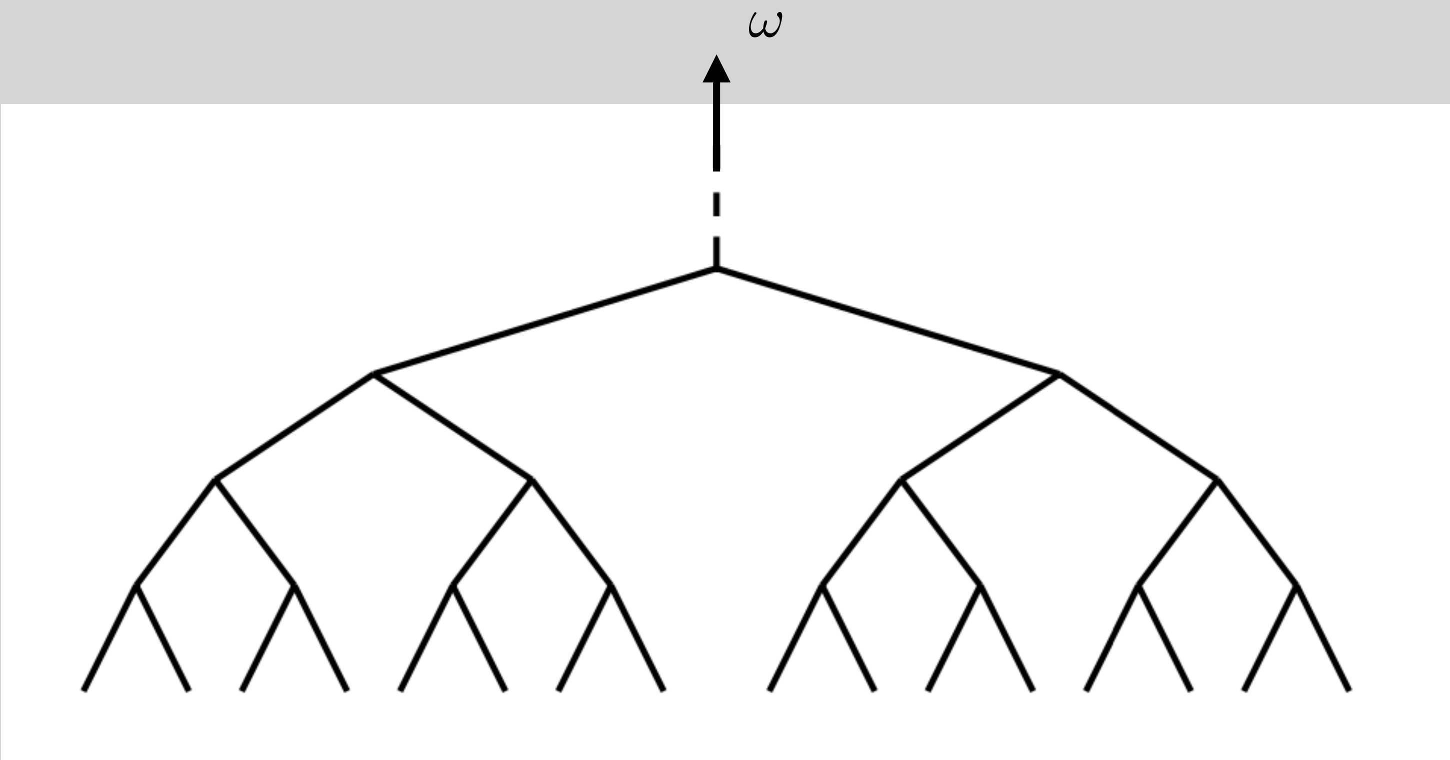

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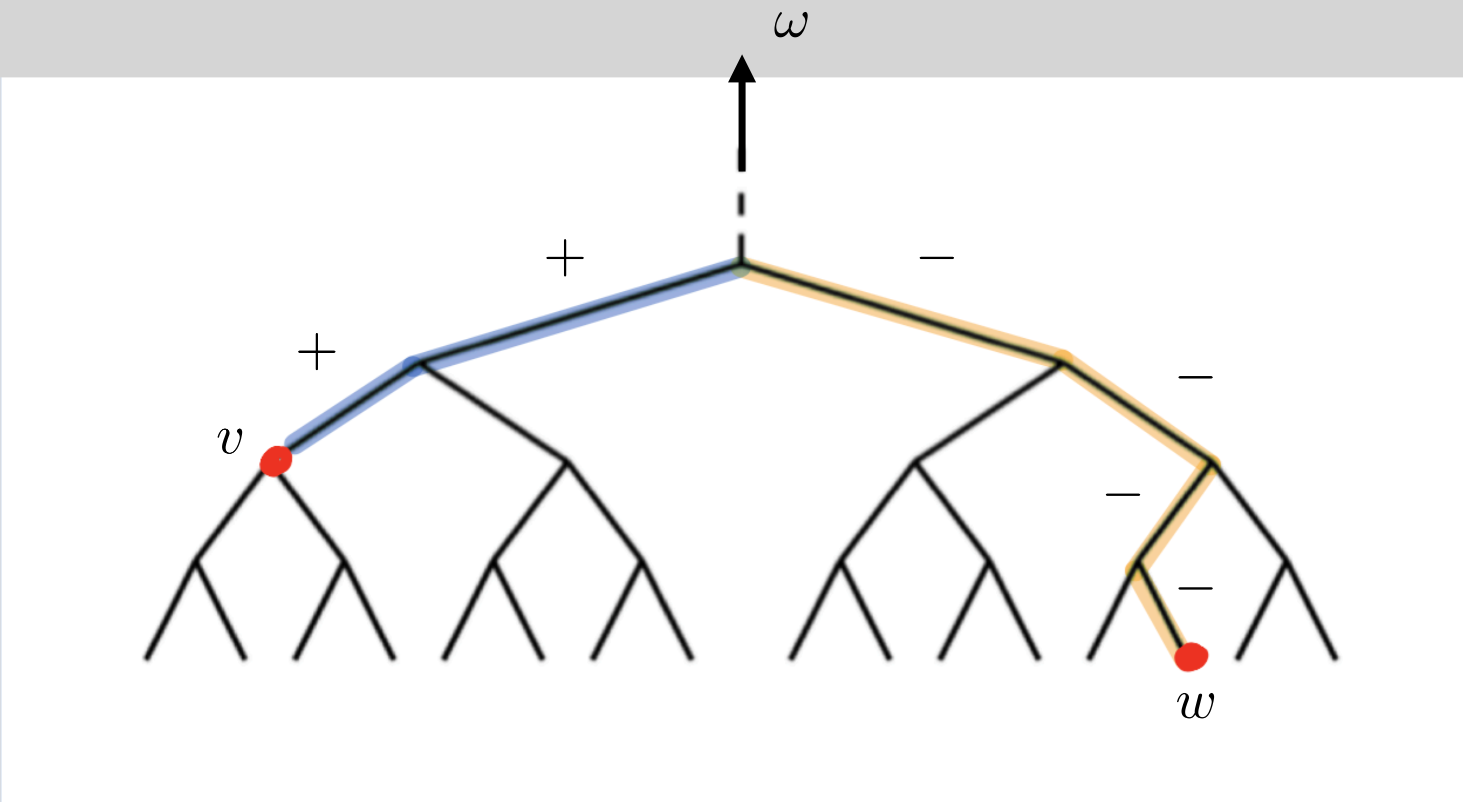

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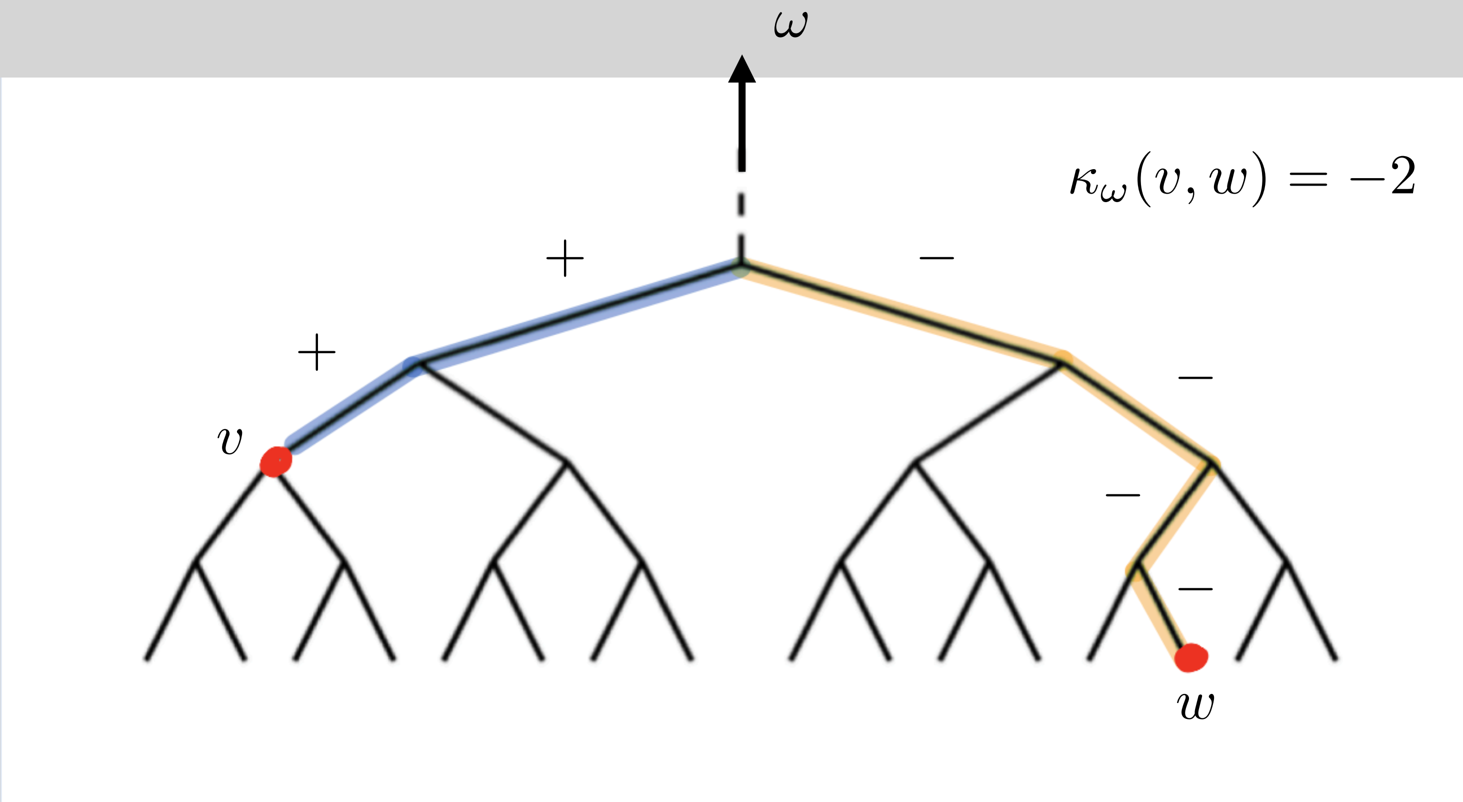

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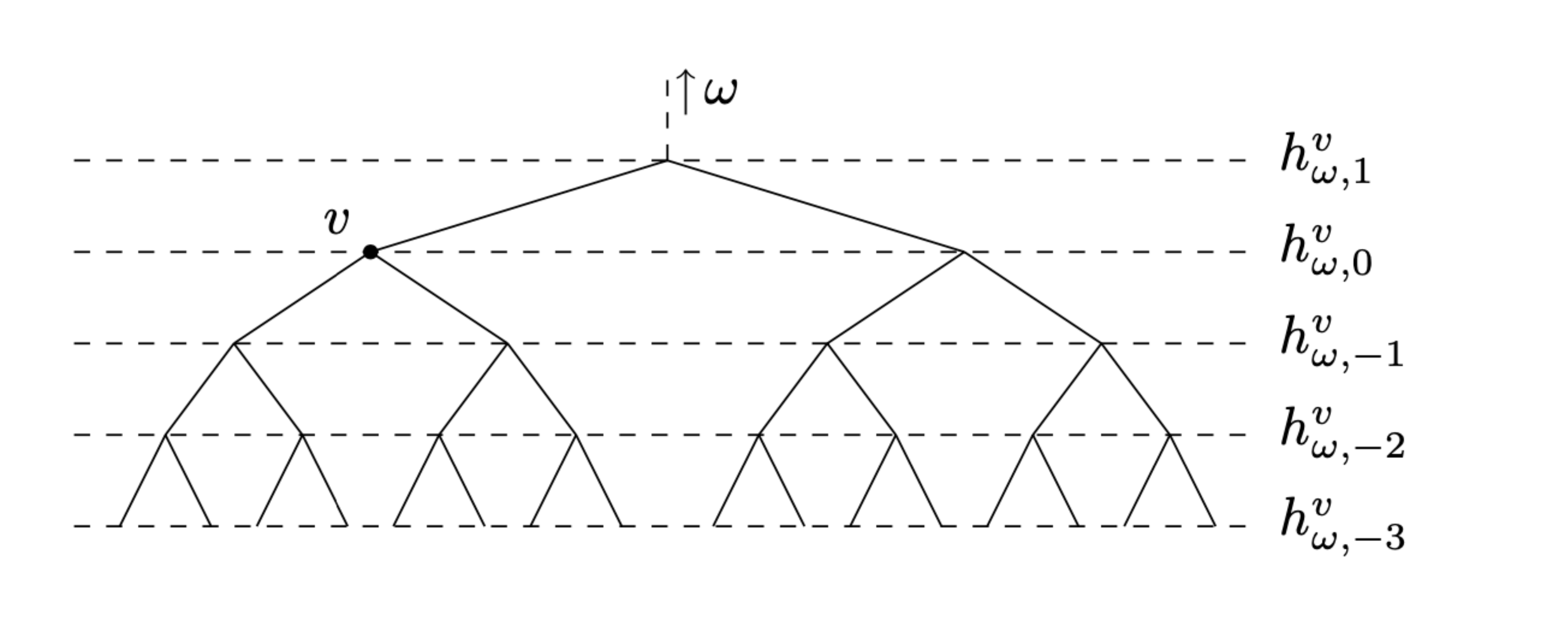

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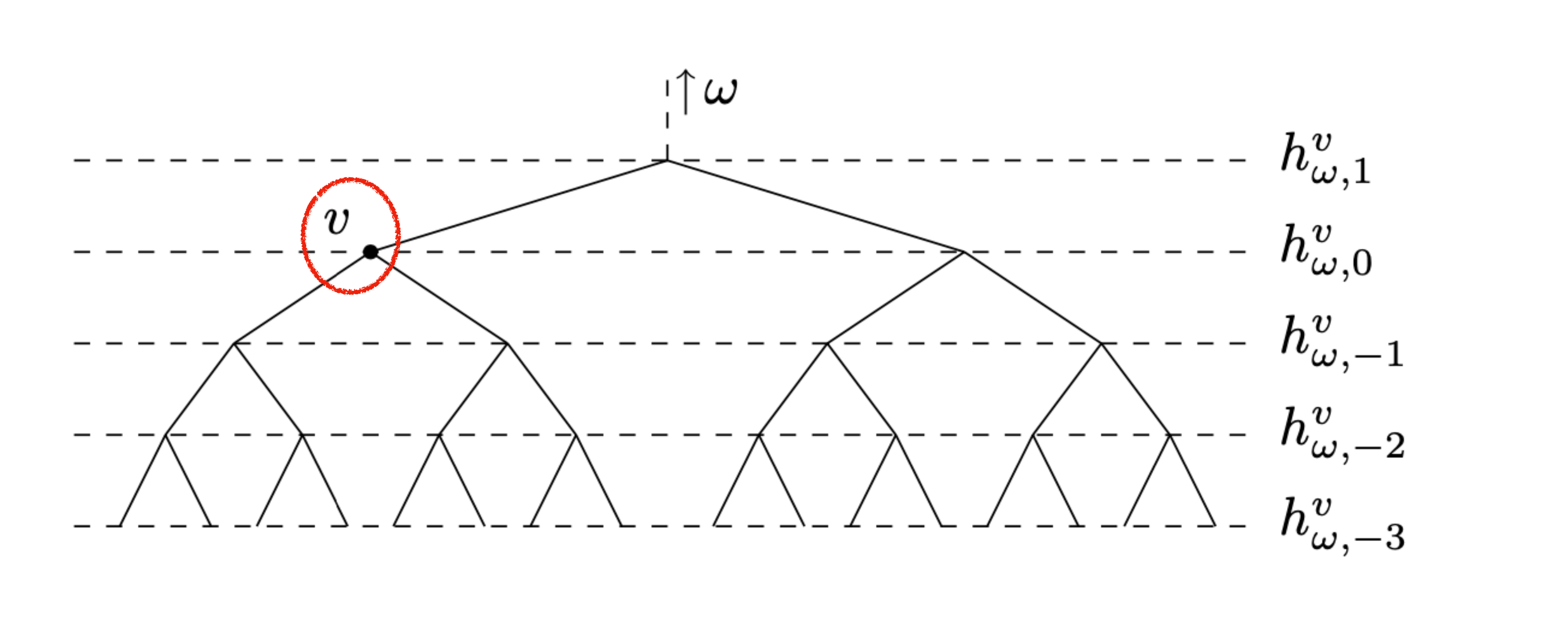

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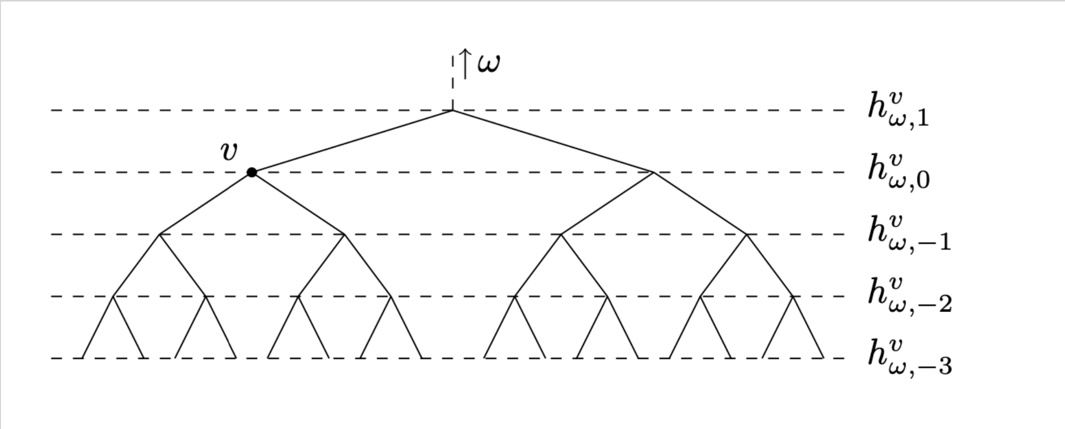

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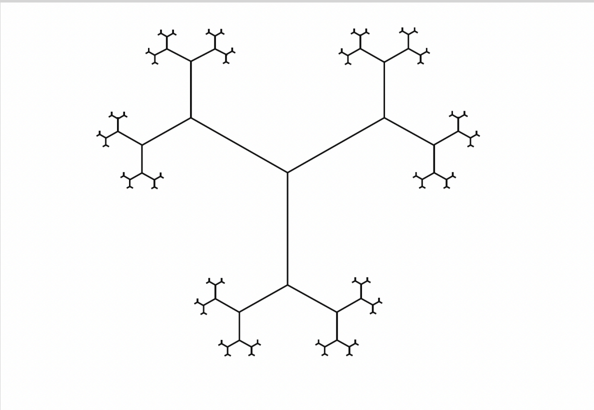

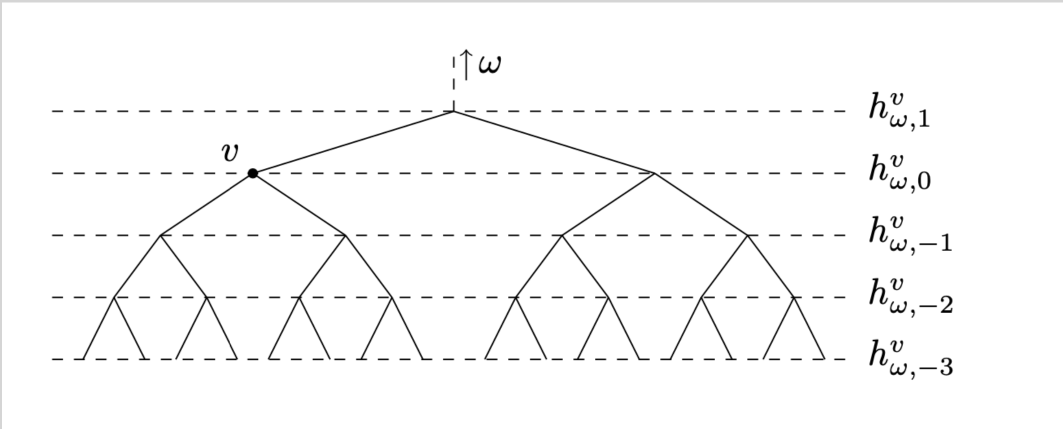

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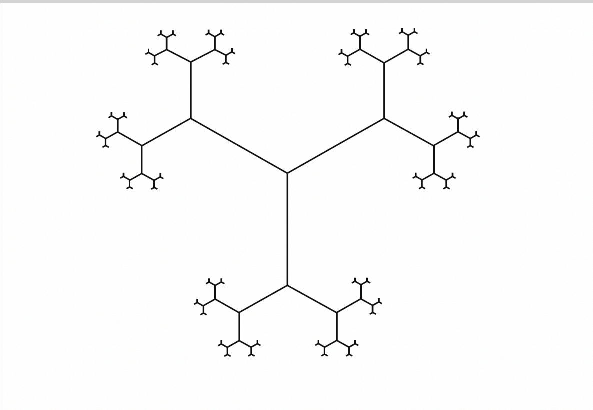

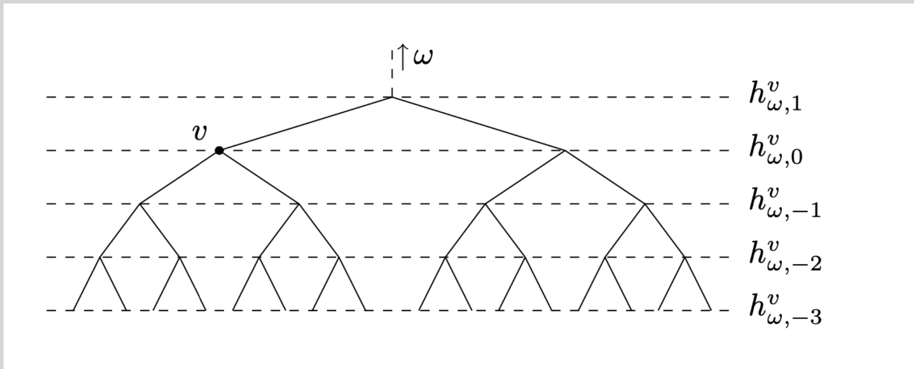

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- $L^2_\flat(\Xi) = \{F \in L^2(\Xi) : F \text{ satisfies property } \flat\}$

the unitarization theorem

- $\Delta^{\frac{1}{2}}(\omega, n) = q^{\frac{n}{2}}$
- $F_v(\omega, n) = F(h^v_{\omega,n}) \leftrightarrow$ identify functions on Ξ and functions on $\Omega \times \mathbb{Z}$
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- $\mathbb{T} = \mathbb{R}/\dfrac{\pi}{\log q}\mathbb{Z}$
- $c_q = \dfrac{q \log q}{4\pi(q+1)}$

the unitarization theorem

Theorem (Bartolucci, Monti & FDM) Let $\mathcal{J}\colon \mathcal{S} \to \mathcal{S}$ be the operator defined by

$$(I \otimes \mathcal{F})(\mathcal{J}\Psi)(\omega, t) = \frac{\sqrt{c_q}}{|\mathbf{c}(\frac{1}{2} + it)|}(I \otimes \mathcal{F})\Psi(\omega, t) \quad (\omega, t) \in \Omega \times \mathbb{Z}$$

and define the operator $\Lambda\colon \mathcal{S}_\flat^\Delta \to \mathcal{S}_\flat^\Delta$ by:

$$(\Lambda F)_v = \Delta^{-1/2}\mathcal{J}(\Delta^{1/2}F_v), \quad v \in X.$$

The composite operator $\Lambda\mathcal{R}$ extends to a unitary operator $\mathcal{Q}\colon L^2(X) \to L_\flat^2(\Xi)$ which satisfies:

$$\widehat{\pi}(g)\mathcal{Q}f = \mathcal{Q}(\pi(g)f), \qquad g \in G,\ f \in L^2(G).$$

F. Bartolucci, M. Monti & FDM , **Unitarization of the Radon Transform on Homogeneous Trees.** *Jour. Fourier Anal. Appl.,* 2021.

the unitarization theorem

critical line

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Thank you