Local smoothing for the wave equation in $2 + 1$ dimensions

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Contents

Local smoothing for the wave equation

Free wave equation in $\mathbb{R}^{n+1}$:

$$\begin{cases} u_{tt} = \Delta_x(u), \\ u|_{t=0} = g, u_t|_{t=0} = 0. \end{cases} \tag{1}$$

Question

If we assume $g \in L^p$, is it always true that $u(\cdot, 1) \in L^p$? If not, how many L^p derivatives it might lose?

- Similar story if we assume generally $u_t|_{t=0} = \tilde{g}$.
- True for $p = 2$.
- If $p \neq 2$, $g \in L^p(\mathbb{R}^n) \not\Rightarrow u(\cdot, 1) \in L^p(\mathbb{R}^n)$: A wave concentrating towards a point can blow up the L^p norm at $t = 1$ ($p > 2$).
- When $p > 2$, one can lose $(n-1)(\frac{1}{2} - \frac{1}{p})$ many derivatives (Seeger-Sogge-Stein) but not more (Peral, also Miyachi).

Sogge's local smoothing conjecture

- If $p > 2$, $g \in L^p(\mathbb{R}^n)$ would not guarantee $u(\cdot, 1) \in L^p(\mathbb{R}^n)$ and one might lose $(n-1)(\frac{1}{2} - \frac{1}{p})$ many derivatives.
- The situation changes if one averages on a local time interval!

Conjecture (Local smoothing (Sogge, 1991))

Let $p = \frac{2n}{n-1}$. If u solves (1), then

$$\|u\|_{L^p(\mathbb{R}^n \times [1,2])} \leq C_\varepsilon \|g\|_{W^{\varepsilon,p}(\mathbb{R}^n)}, \forall \varepsilon > 0.$$

Here $W^{\varepsilon,p}$ is the Sobolev space consist of functions in L^p with "ε-many L^p derivatives".

- "The propagator of the wave equation is almost bounded on $L^p, 2 \leq p \leq \frac{2n}{n-1}$ if one averages locally in time".

Local smoothing in $2 + 1$ dimensions

In the remaining time we fix $n = 2$.

Conjecture (Local smoothing in $2 + 1$ dimensions)

If u solves (1), then

$$\|u\|_{L^4(\mathbb{R}^2\times[1,2])} \leq C_\varepsilon \|g\|_{W^{\varepsilon,4}(\mathbb{R}^2)}, \forall \varepsilon > 0.$$

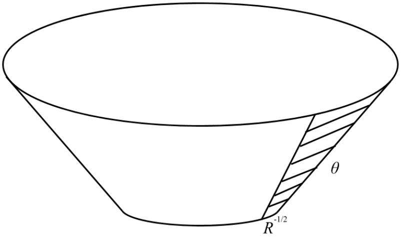
- Sogge proved some nontrivial local smoothing.
- Mockenhaupt-Seeger-Sogge: ε can be $\frac{1}{8}$.
- Improved by Tao-Vargas, Wolff, Lee.
- The conjecture is true for $n = 2$ (Guth-Wang-Z., 2020).

Wolff's decoupling approach

There is a local smoothing conjecture for $p > 4$, asserting you almost gain $1/p$ derivatives when averaged over the time interval $[1, 2]$.

- Wolff invented ($l^p L^p$) decoupling (2000) to prove the above conjecture for $p \geq 74$.
- Let $\Gamma = \{|\xi_3| = |(\xi_1, \xi_2)|, 1 \leq \xi_3 \leq 2\}$ be the truncated cone and partition it into sectors θ of aperture $R^{-1/2}$ for some large parameter R.
- Assuming supp$\hat{f} \subset \Gamma$. Consider the decomposition $f = \sum_\theta f_\theta$ where $\widehat{f_\theta}$ is the restriction of $\widehat{f}$ on θ.
- We state Wolff's result on the next slide.

Results from decoupling



Theorem (Wolff)

For $p \geq 74$, $\|f\|_p \lesssim_\varepsilon R^{\frac{1}{2}-\frac{2}{p}+\varepsilon}(\sum_\theta \|f_\theta\|_p^p)^{\frac{1}{p}}$.

- Bourgain-Demeter proved that the above theorem is true for $p \geq 6$. This range is optimal.
- Such theorems imply the corresponding L^p local smoothing.
- The range $4 \leq p < 6$ for local smoothing cannot be covered by decoupling.

The square function estimate

In our paper, we are able to prove a different result:

Theorem (Square function estimate, Guth-Wang-Z.)

$$\|f\|_4 \lesssim_\varepsilon R^\varepsilon \|(\sum_\theta |f_\theta|^2)^{\frac{1}{2}}\|_4.$$

- It implies Sogge's local smoothing conjecture (in $2+1$ dimensions).
- The proof does not rely on decoupling, but was inspired a lot by Bourgain-Demeter's proof of the decoupling conjecture.

Ideas in the proof: Localization

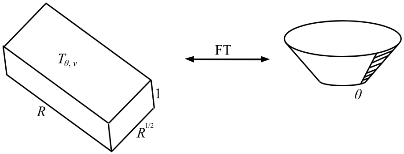
By rescaling, Littlewood-Paley and a localization reduction, we will always assume $\mathrm{supp}\hat{f} \subset \Gamma_{\frac{1}{R}}$ (the $\frac{1}{R}$-neighborhood of Γ) and try to prove

$$\frac{1}{R}\|f\|_{L^4(B_R^3)}^4 \lesssim_\varepsilon R^\varepsilon \|f(\cdot,0)\|_{L^4(\mathbb{R}^2)}^4.$$

Here B_R^3 is the R-ball around 0 in $\mathbb{R}^3$.
We will change the meaning of θ to its convex hull (or its $\frac{1}{R}$-neighborhood), roughly a box of dimensions $1 \times R^{-\frac{1}{2}} \times R^{-1}$.

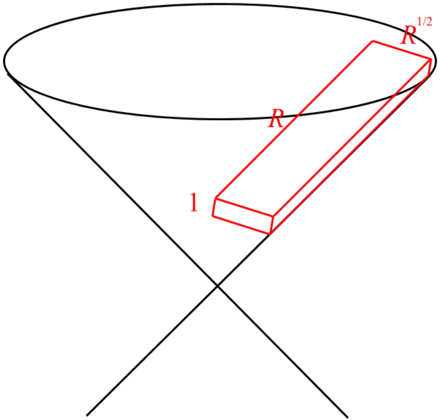
Ideas in the proof: Wave packet decomposition

We can decompose f and f_θ further into pieces that essentially localize both in the physical space and in the frequency space (Fefferman, Bourgain, ...).



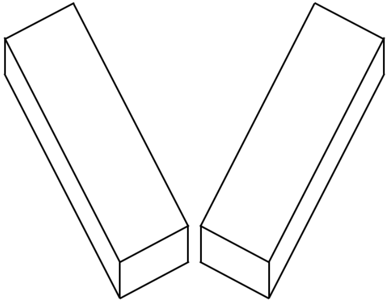
- $f = \sum_{\theta,v} f_{\theta,v}$.
- $f_{\theta,v}$: Wave packets.
- Each $f_{\theta,v}$ has frequency support in (the box) θ and physical support essentially in $T_{\theta,v}$ (direction dual to θ).
- $|f_{\theta,v}| \approx$ const (called *magnitude* next) on $T_{\theta,v}$.

Boxes adapted to the geometry of the light cone



Ideas in the proof: Case studies (1)

Assuming our f is a sum of N wave packets and each wave packet has magnitude ~ 1, what is a good upper bound of $\|f\|_{L^4(B_R)}^4$?

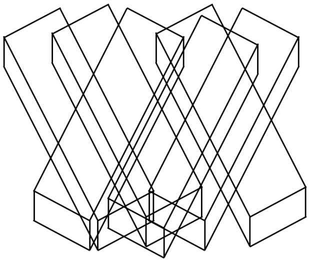


Example 1: Separated wave packets case

When wave packets have separated supports, $\|f\|_{L^4(B_R)}^4 \sim NR^{\frac{3}{2}}$.

Ideas in the proof: Case studies (2)

Assuming our f is a sum of N wave packets and each wave packet has magnitude ~ 1, what is a good upper bound of $\|f\|_{L^4(B_R)}^4$?



Example 2: A Random example

When wave packets have random positions and signs and $N \geq R^{\frac{3}{2}} = \frac{|B_R|}{|T_{\theta,v}|}$, we expect $\|f\|_{L^4(B_R)}^4 \sim R^3(NR^{-\frac{3}{2}})^{\frac{1}{2}\cdot 4} = N^2$.

Ideas in the proof: Case studies (cont.)

- Assuming our f is a sum of N wave packets and each wave packet has magnitude ~ 1, what is a good upper bound of $\|f\|_{L^4(B_R)}^4$?
- Based on the previous two examples, it is tempting to guess

$$\|f\|_{L^4(B_R)}^4 \lesssim_\varepsilon R^\varepsilon (NR^{\frac{3}{2}} + N^2).$$

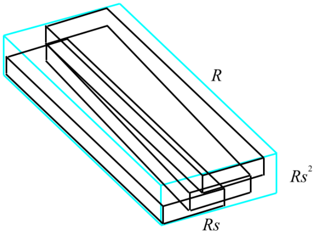
Ideas in the proof: Case studies (cont.)

- Assuming our f is a sum of N wave packets and each wave packet has magnitude ~ 1, what is a good upper bound of $\|f\|_{L^4(B_R)}^4$?
- Based on the previous two examples, it is tempting to guess

$$\|f\|_{L^4(B_R)}^4 \lesssim_\varepsilon R^\varepsilon (NR^{\frac{3}{2}} + N^2).$$

- Unfortunately, this is wrong!

Ideas in the proof: Case studies (3)



Example 3: A Random example in one fat plank

- Consider a fat plank U of dimensions $Rs^2 \times Rs \times R$ with the direction along the light cone. Here $R^{-\frac{1}{2}} \leq s \leq 1$.
- When all wave packets are in U and they have random positions and signs and $N \geq s^3 R^{\frac{3}{2}} = \frac{|U|}{|T_{\theta,v}|}$, we expect $\|f\|_{L^4(B_R)}^4 \sim |U|(Ns^{-3}R^{-\frac{3}{2}})^{\frac{1}{2}\cdot 4} = s^{-3}N^2$.

Ideas in the proof: A conjecture based on case studies

For each dyadic $s \in [R^{-\frac{1}{2}}, 1]$, we decompose $\Gamma_{\frac{1}{R}}$ into sectors τ of aperture $d(\tau) = s$. For each τ, we take all $\theta \subset \tau$ and take U_τ to be the convex hull of all θ^* with $\theta \subseteq \tau$. The "fat plank" U_τ has dimensions $Rs^2 \times Rs \times R$.

For each U_τ, tile B_R by translations $U \,/\!/\, U_\tau$. For each U, let f_U be the sum of all wave packets that are essentially supported on U.

Note: each such wave packet must have frequency support in τ.

Conjecture

$$\|f\|_{L^4(B_R)}^4 \lesssim_\varepsilon R^\varepsilon \sum_{\text{dyadic } s \in [R^{-\frac{1}{2}}, 1]} \sum_{d(\tau)=s} \sum_{U /\!/ U_\tau} |U|^{-1} \|f_U\|_{L^2}^4.$$

- The conjecture is essentially true and it essentially implies the square function conjecture and the local smoothing conjecture.

Ideas in the proof: Can we induct on scales?

In the remaining time, we focus on the conjecture on the last slide. We want to prove

$$\|f\|_{L^4(B_R)}^4 \lesssim_\varepsilon R^\varepsilon \sum_{\text{dyadic } s\in[R^{-\frac{1}{2}},1]} \sum_{d(\tau)=s} \sum_{U /\!/ U_\tau} |U|^{-1} \|f_U\|_{L^2}^4.$$

- Does it help if we already know

$$\|f\|_{L^4(B_r)}^4 \lesssim_\varepsilon r^\varepsilon \sum_{\text{dyadic } s\in[r^{-\frac{1}{2}},1]} \sum_{d(\widetilde{\tau})=s} \sum_{U /\!/ U_{\widetilde{\tau}}} |U|^{-1} \|f_U\|_{L^2}^4$$

  for some $r \leq R/2$?
- Comparing (RHS) of the above inequalities and using Lorentz rescaling, we see it makes sense to be interested in a more general class of inequalities.

Ideas in the proof: Introducing $S(r, R)$

Given dyadic $1 \le r \le R$. Define $S(r, R)$ to be the best constant such that

$$\sum_{B_r \subset B_R} |B_r|^{-1} \|f\|_{L^2(B_r)}^4 \le S(r, R) \sum_{\text{dyadic } s \in [R^{-\frac{1}{2}}, 1]} \sum_{d(\tau)=s} \sum_{U // U_\tau} |U|^{-1} \|f_U\|_{L^2}^4.$$

- "$\sum_{B_r \subset B_R}$" is over a finitely overlapping collection.
- We want $S(1, R) \lesssim_\varepsilon R^\varepsilon$. Which is morally true!
- By Lorentz rescaling, it is not hard to show

$$S(r_1, r_3) \lesssim \log r_2 \cdot S(r_1, r_2) \cdot \max_{r_2^{-\frac{1}{2}} \le s \le 1} S(s^2 r_2, s^2 r_3)$$

where $r_1 \le r_2 \le r_3$ and s are dyadic.

Ideas in the proof: Missing pieces of the puzzle

- We want to show $S(1,R) \lesssim_\varepsilon R^\varepsilon$.
- We already know

$$S(r_1,r_3) \lesssim \log r_2 \cdot S(r_1,r_2) \cdot \max_{r_2^{-\frac{1}{2}} \le s \le 1} S(s^2 r_2, s^2 r_3).$$

- $S(R^{\frac{1}{2}}, R)$ is "friendlier" to deal with. Indeed we morally proved $S(r_1,R) \lesssim 1$ whenever $R^{\frac{1}{2}} \le r_1 \le R$. It was proved very nontrivially by new techniques in incidence geometry.
- What is left? We are done as long as we can show $S(1,K) \le K^\varepsilon$ for one fixed constant $K = K(\varepsilon)$ that is large enough (for technical reasons).

Ideas in the proof: Bounding $S(R^{\frac{1}{2}}, R)$

- First we explain how to (morally) prove $S(R^{\frac{1}{2}}, R) \lesssim 1$.
- By orthogonality of wave packets and Hölder, it suffices to prove

$$\int_{B_R} (\sum_{\theta,v} |f_{\theta,v}|^2)^2$$
$$\lesssim \sum_s \sum_{d(\tau)=s} \sum_{U /\!/ U_\tau} |U|^{-1} (\int_U \sum_{(\theta,v):\ \text{ess. supp. of } f_{\theta,v} \subset 10U} |f_{\theta,v}|^2)^2.$$

- Morally we prove this inequality by an incidence theorem.

Ideas in the proof: Bounding $S(R^{\frac{1}{2}}, R)$ using incidence geometry

Theorem (Essentially in [GWZ])

Let $\{T\}$ be a collection of $1 \times R^{\frac{1}{2}} \times R$-planks with directions along the light cone. Then

$$\int_{B_R} (\sum 1_T)^2 \lesssim_\varepsilon R^\varepsilon \sum_{s \in [R^{-\frac{1}{2}}, 1]} \sum_{d(\tau)=s} \sum_{U /\!/ U_\tau} |U|^{-1} (\int_U \sum_{T /\!/ \text{ some } \theta^* \text{ with } \theta \subset \tau} 1_T)^2$$

where $U \subset 10B_R$.

Ideas in the proof: Bounding $S(R^{\frac{1}{2}}, R)$ using incidence geometry

- For the parabola there is an analogous theorem which can be proved elementarily.
- Method of the proof: viewing 1_T roughly as a smooth bump function ψ_T and then studying the Fourier transform of $\sum_T \psi_T$ via a high/low frequency decomposition.
- Based on the work of Orponen, Vinh, Guth-Solomon-Wang and Demeter-Guth-Wang.
- The geometry of the cone is used in an essential way.

Ideas in the proof: Bounding $S(1, K)$

- It remains to find a large enough (for technical reasons) constant K such that we have $S(1, K) \leq K^{\varepsilon}$.
- To achieve this, we realize that it suffices to have a K where a similar inequality to $S(1, K) \leq K^{\varepsilon}$ holds when we further assume supp$\hat{f}$ is in a shortened cone of height $\frac{1}{K}$.
- This is then confirmed by estimates for the parabola.
- A similar trick was used in Bourgain-Demeter's proof of the Decoupling Conjecture. See also Pramanik-Seeger.

Thank you!