Real algebraic geometry, and the Kakeya and Restriction conjectures

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A *Besicovitch set* is a compact set $K \subset \mathbb{R}^n$ that contains a unit line segment pointing in every direction.

Kakeya set conjecture:
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Kakeya set conjecture:
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- Proved for $n = 2$ (Davies 1971).
- Open for $n \geq 3$, though partial progress.

Kakeya maximal function conjecture:
For $\delta > 0$, let $\mathbb{T}$ be a set of δ tubes in $\mathbb{R}^n$ that point in δ-separated directions. Then for each $1 \le d \le n$,

$$\Big\| \sum_{T \in \mathbb{T}} \chi_T \Big\|_{\frac{d}{d-1}} \le C(n, \varepsilon) \Big(\frac{1}{\delta}\Big)^{\frac{n}{d} - 1 + \varepsilon}. \tag{1}$$

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- Proved for $n = 2$ (Cordoba 1977).
- Open for $n \geq 3$, though partial progress.

If (1) holds for some $d \leq n$, we say a Kakeya maximal function estimate holds in $\mathbb{R}^n$ at dimension d.

Progress towards the Kakeya maximal function conjecture.

n	d	
2	2	Cordoba 1977
$n \geq 3$	$(n+1)/2$	Christ-Duoandikoetxea-Rubio de Francia 1987
$n \geq 3$	$(n+1)/2 + c_n$	Bourgain 1991
$n \geq 3$	$(n+2)/2$	Wolff 1995
$n \gg 1$	$(1+\varepsilon)n/2$	Bourgain 1999
$n \geq 9$	$(4n+3)/7$	Katz-Tao 2002
4	$3 + 1/40$	Guth-Z. 2017; Z. 2018
3	$5/2 + \varepsilon$	Katz-Z. 2019
$n \subset [5, 97]$	$> (4n+3)/7$	Hickman-Rogers 2020
4	3.059	Katz-Z. 2021
$n = 5, \geq 7$	$(2-\sqrt{2})n + c_n$	Hickman-Rogers-Zhang; Z. 2021

Kakeya numerology

Every Besicovitch set in $\mathbb{R}^n$ has Hausdorff dimension $\geq d$

$$\Updownarrow$$

Kakeya maximal function estimate in $\mathbb{R}^n$ holds at dimension d.

$$\Updownarrow$$

For a typical $x \in \bigcup_{\mathbb{T}} T$, there are $\leq \delta^{d-n}$ tubes containing x.

Multilinear Kakeya (Bennett-Carbery-Tao 2006, Guth 2010):

Let $\mathbb{T}_1, \ldots, \mathbb{T}_n$ be sets of δ tubes in $\mathbb{R}^n$, $\#\mathbb{T}_i \leq \delta^{1-n}$.
Suppose the tubes in $\mathbb{T}_i$ make large angle with the tubes in $\mathbb{T}_j$.

Then

$$\left\| \Big(\sum_{T_1 \in \mathbb{T}_1} \sum_{T_2 \in \mathbb{T}_2} \cdots \sum_{T_n \in \mathbb{T}_n} \chi_{T_1} \chi_{T_n} \cdots \chi_{T_n} \Big)^{\frac{1}{n}} \right\|_{\frac{n}{n-1}} \leq C(n).$$

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A slightly more complicated version (Carbery-Valdimarsson):

Let $\mathbb{T}_1, \ldots, \mathbb{T}_k$ be sets of δ tubes in $\mathbb{R}^n$, $2 \leq k \leq n$. Then

$$\Big\| \Big(\sum_{T_1 \in \mathbb{T}_1} \cdots \sum_{T_k \in \mathbb{T}_k} \chi_{T_1} \cdots \chi_{T_n} |v_1 \wedge \ldots \wedge v_k| \Big)^{\frac{1}{k}} \Big\|_{\frac{k}{k-1}}$$

$$\leq C(n) \Big(\frac{1}{\delta} \Big)^{\frac{n}{k}-1} \prod_{i=1}^{k} (\delta^{n-1} \#\mathbb{T}_i)^{\frac{1}{k}}.$$

Moral:

Let $\mathbb{T}$ be a set of distinct δ tubes in $\mathbb{R}^n$, with $\#\mathbb{T} \leq \delta^{1-n}$.

Suppose that for a typical $x \in \bigcup_{\mathbb{T}} T$, most k-tuples of tubes containing x satisfy $|v_1 \wedge \ldots \wedge v_k| \sim 1$.
Then a typical x is contained in $\leq \delta^{k-n}$ tubes.

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Opposite extreme:

Suppose that for a typical $x \in \bigcup_{\mathbb{T}} T$, the tubes containing x are in the δ-neighborhood of a $k-1$ plane.
Then a typical x is contained in $\leq \delta^{2-k}$ tubes.

The Bourgain-Guth multilinear $\to$ linear machine

$$
\begin{tikzcd}[column sep=small]
 & \begin{array}{c}\mathbb{T}\text{: } \delta \text{ tubes in } \mathbb{R}^n \\ 2 \leq k \leq n\end{array} \arrow[dl, "k\text{-transverse}"'] \arrow[dr, "(k-1)\text{-plainy}"] & \\
\begin{array}{c}\mathbb{T} \text{ satisfies (1)} \\ \text{for } d = k\end{array} \arrow[dr] & & \begin{array}{c}\mathbb{T} \text{ satisfies (1)} \\ \text{for } d = n - k + 2\end{array} \arrow[dl] \\
 & \begin{array}{c}\mathbb{T} \text{ satisfies (1)} \\ \text{for } d = \min(k, n - k + 2)\end{array} &
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\end{tikzcd}$$

Conclusion:

A Kakeya maximal function estimate holds at dimension

$$d = \begin{cases} (n+2)/2, & n \text{ even}, \\ (n+1)/2, & n \text{ odd} \end{cases}$$

Improved multilinear Kakeya for direction-separated tubes (Z. 2021)

Let $\mathbb{T}_1, \ldots, \mathbb{T}_k$ be sets of δ tubes that point in δ-separated directions. Then

$$\left\|\Big(\sum_{T_1 \in \mathbb{T}_1} \cdots \sum_{T_k \in \mathbb{T}_k} \chi_{T_1} \ldots \chi_{T_k} |v_1 \wedge \ldots \wedge v_k|^{\frac{k}{d}}\Big)^{\frac{1}{k}}\right\|_{\frac{d}{d-1}} \leq C(n, \varepsilon)\Big(\frac{1}{\delta}\Big)^{\frac{n}{d}-1-\varepsilon},$$

$$d = \frac{n^2 + k^2 + n - k}{2n}.$$

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Hickman-Rogers-Zhang concurrently proved a similar result.

Corollary (Hickman-Rogers-Zhang; Z. 2021)

A Kakeya maximal function estimate in $\mathbb{R}^n$ holds at dimension $d = (2 - \sqrt{2})n + c_n$, $c_n \geq 0$.

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A Kakeya maximal function estimate in $\mathbb{R}^n$ holds at dimension $d = (2 - \sqrt{2})n + c_n$, $c_n \geq 0$.

This is new in dimension $n \geq 7$ (and $n = 5$).

Also improves the Kakeya set conjecture for certain n.

How do we prove this theorem? Basic idea:

- The function we want to estimate is supported on $E = \bigcap_{i=1}^{k} \bigcup_{T \in \mathbb{T}_i} T$.
- We break E into pieces, so that few tubes from $\mathbb{T}_1, \ldots, \mathbb{T}_k$ intersect each piece.
- We apply (conventional) k-linear Kakeya to estimate the L^d norm on each piece.

To make this work, we need a way to partition E, so that few tubes intersect each piece.

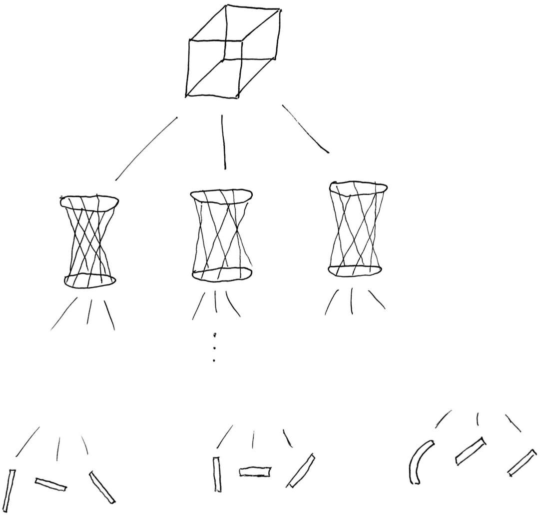
We do this using polynomial partitioning.

Polynomial partitioning (Guth-Katz 2010): Basic numerology

Given a parameter $t \geq 1$, we can partition $\mathbb{R}^m$ (or a low degree m-dimensional variety) into about t^m cells, plus a boundary, so that each tube intersects $O(t)$ cells.

The boundary is the δ-neighbourhood of a variety. We iterate this process and construct a tree of grains.

Tree of grains



j-th level: co-dimension j grains; diameter λ_j; branching D_{n-j}^{n-j}

The main enemy

- Applying k-linear Kakeya is effective if the tree has many leaves.
- What if the tree has few leaves?
- Then each grain in the tree has large intersection with many tubes from $\mathbb{T}_i$, $i = 1, \ldots, k$.

The main enemy

- Applying k-linear Kakeya is effective if the tree has many leaves.
- What if the tree has few leaves?
- Then each grain in the tree has large intersection with many tubes from $\mathbb{T}_i$, $i = 1, \ldots, k$.
- If the varieties in the tree are linear (i.e. the grains are rectangular prisms), then this cannot happen too much, since the tubes in $\mathbb{T}_i$ point in δ-separated directions.
- What about more general varieties?

Theorem (Katz-Rogers 2018)
Let $\mathbb{T}$ be a set of δ tubes pointing in δ-separated directions.
Let $\delta \leq \lambda \leq 1$.
Let $S \subset \mathbb{R}^n$ be a semi-algebraic set. Then

$$\#\{T \in \mathbb{T} \colon |T \cap S| \geq \lambda |T|\} \lessapprox \frac{\delta^{1-n}|S|}{\lambda^n}.$$

(The RHS is what you expect if S is a rectangular prism.)

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Direction-separated tubes obey the polynomial Wolff axioms.

Theorem (Hickman-Rogers-Zhang; Z.)
Let $\mathbb{T}$ be a set of δ tubes pointing in δ-separated directions.
Let $S_1 \supset S_2 \supset \cdots \supset S_k$ be nested sequence of grains (δ-neighborhoods of varieties), where S_i has codimension $\geq i$ and diameter λ_i. Then

$$\#\{T \in \mathbb{T} \colon |T \cap S_i| \geq \lambda_i |T|,\ i = 1, \ldots, k\} \lessapprox \frac{\delta^{k+1-n}}{\lambda_1 \cdots \lambda_k}.$$

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Direction-separated tubes obey the nested polynomial Wolff axioms.

How do we prove this?
The answer is related to a semi-algebraic version of Keleti's line segment extension conjecture.

Proposition
Let $S \subset \mathbb{R}^n$ be a semi-algebraic set and let $\mathbb{T}$ be the set of $\lambda \times \delta$ tubes contained in S. If we lengthen each of these tubes by a factor of $1 \leq A \leq \lambda^{-1}$, then their union has volume $\lessapprox A^n|S|$.

Semi-algebraic Keleti

$\Downarrow$

Nested polynomial Wolff axioms

$\Downarrow$

Tree of grains has many leaves.

$\Downarrow$ (k-linear Kakeya)

Improved k-linear Kakeya for direction-separated tubes

$\Downarrow$ (Bourgain-Guth machine)

Kakeya maximal function estimate

Semi-algebraic Keleti

$$\Downarrow$$

Nested polynomial Wolff axioms

$$\Downarrow$$

Tree of grains has many leaves.

$$\Downarrow \quad \text{(}k\text{-linear Restriction)}$$

Improved k-broad restriction

$$\Downarrow \quad \text{(Bourgain-Guth machine)}$$

Fourier restriction estimate

Fourier restriction

The Fourier extension operator for the truncated paraboloid:

$$Ef(x) = \int_{B^{n-1}} f(\omega)\, e^{i(x_1\omega_1+\cdots+x_{n-1}\omega_{n-1}+x_n|\omega|^2)}\,\mathrm{d}\omega,$$

where B^{n-1} is the unit ball in $\mathbb{R}^{n-1}$.

Stein conjectured that

$$\|Ef\|_{L^p(\mathbb{R}^n)} \leq C_{n,p}\|f\|_{L^p(B^{n-1})} \qquad (\mathrm{R}_p^*)$$

holds for all $p > \frac{2n}{n-1}$.

The conjecture has been proved for $n = 2$ (Fefferman-Stein 1970).

It remains open for $n \geq 3$, though partial progress has been made by many researchers, including Bourgain, Fefferman, Guth, Hickman, Moyua, Rogers, Stein, Tao, Tomas, Vargas, Vega, Wang, Wolff, Zhang, Zygmund.

For n large, Stein's conjecture says the extension operator E is bounded on L^p for $p > 2 + 2n^{-1} + O(n^{-2})$.

Currently, we know that E is bounded on L^p for $p > 2 + \lambda n^{-1} + O(n^{-2})$, where

λ	
4	Tomas 1975
3	Bourgain-Guth 2011
$8/3$	Guth 2018
$2.604\ldots$	Hickman-Rogers 2019
$2.596\ldots$	Hickman-Z. 2022

Thanks for listening.